Quiz 1

(Supplementary)

24th September, 2025

Time: 2 hrs **Total marks:** 26

On the real line $\mathbb{R}$, let $\mathcal{T}_{\geq}$ be the collection of subsets consisting of $\emptyset$, along with the usual open sets $U \subset \mathbb{R}$ satisfying

$$\mathbb{Z}_{\geq n} := \{n, n+1, n+2, \ldots\} \subset U, \text{ for some } n \in \mathbb{Z}.$$

Attempt any question. You can get **maximum 20**.

Q1. Show that $\mathcal{T}_{\geq}$ is a topology on $\mathbb{R}$. [2]

Q2. Compare (i.e., strictly fine, strictly coarse or incomparable) $\mathcal{T}_{\geq}$ with the following. $[1 \times 4 = 4]$

i) The usual topology on $\mathbb{R}$.

ii) The lower limit topology $\mathbb{R}_l$.

iii) The upper limit topology $\mathbb{R}_u$.

iv) The topology $\mathcal{T}_{\rightarrow} = \{\emptyset, \mathbb{R}\} \bigcup \{(a, \infty) \mid a \in \mathbb{R}\}$ on $\mathbb{R}$.

Q3. For $a \in \mathbb{R}$, determine (with justification) the closures of the following sets in $(\mathbb{R}, \mathcal{T}_{\geq})$. $[1 \times 5 = 5]$

i) (a, ∞).

ii) $(-\infty, a)$.

iii) $\{a\}$.

iv) $A = \{a, a+1, a+2, \ldots\}$.

v) $B = \{a, a-1, a-2, \ldots\}$.

Q4. Determine (with justification) whether $(\mathbb{R}, \mathcal{T}_{\geq})$ is T_0, T_1, or T_2. $[1 \times 3 = 3]$

Q5. Prove or give counter-example to the following statements. $[1 \times 2 = 2]$

i) If a sequence (x_n) converges to x in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$, then $x_n \to x$ in $(\mathbb{R}, \mathcal{T}_{\geq})$ as well.

ii) If a sequence (x_n) converges to x in $(\mathbb{R}, \mathcal{T}_{\geq})$, then $x_n \to x$ in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ as well.

Q6. Prove or disprove : $(\mathbb{R}, \mathcal{T}_{\geq})$ is path connected. [2]

Q7. Consider the equivalence relation on $\mathbb{R}$: $a \sim b$ if and only if $a - b \in \mathbb{Z}$. For any $x \in \mathbb{R}$, find the closure of the equivalence class $[x]$ in the quotient topology induced from $(\mathbb{R}, \mathcal{T}_{\geq})$. [4]

Q8. Consider the equivalence relation on $\mathbb{R}$: $a \sim b$ if and only if either

$$a, b \in \mathbb{R} \setminus \mathbb{Z}, \text{ and } a = b, \qquad \text{or}, \qquad a, b \in \mathbb{Z}.$$

For any $x \in \mathbb{R}$, find the closure of the equivalence class $[x]$ in the quotient topology induced from $(\mathbb{R}, \mathcal{T}_{\geq})$. [4]