Quiz 1

11th September, 2025

Time: 2 hrs **Marks:** ____/20

On the real line $\mathbb{R}$, consider the collection of subsets

$$\mathcal{T}_{\rightarrow} := \{\emptyset, \mathbb{R}\} \bigcup \{(a, \infty) \mid a \in \mathbb{R}\}.$$

Attempt any question. You can get **maximum 20**.

Q1. Show that $\mathcal{T}_{\rightarrow}$ is a topology on $\mathbb{R}$.

Solution. Clearly $\emptyset, \mathbb{R} \in \mathcal{T}_{\rightarrow}$. Consider a collection $\{U_\alpha \in \mathcal{T}_{\rightarrow}\}$. If any $U_\alpha = \emptyset$, we can ignore them, and if any $U_\alpha = \mathbb{R}$, then clearly $\bigcup U_\alpha = \mathbb{R} \in \mathcal{T}_{\rightarrow}$. Thus, assume that $U_\alpha = (a_\alpha, \infty)$. Now, for the set $A = \{a_\alpha\} \subset \mathbb{R}$, there are two possibilities.

(a) A is lower bounded. Hence, there is some $a_0 = \inf A$. Now, clearly $\bigcup (a_\alpha, \infty) \subset (a_0, \infty)$, as $a_0 \leq a_\alpha$ for all α. Also, for any $a_0 < x$, by the property of infimum (i.e., greatest lower bound), we have $a_0 \leq a_\alpha < x$ for some $a_\alpha \in A$. But then $x \in (a_\alpha, \infty)$. Consequently, $(a_0, \infty) \subset \bigcup (a_\alpha, \infty)$. Thus, $\bigcup (a_\alpha, \infty) = (a_0, \infty) \in \mathcal{T}_{\rightarrow}$.

(b) A is not lower bounded. Then, $\bigcup (a_\alpha, \infty) = \mathbb{R} \in \mathcal{T}_{\rightarrow}$.

Finally, for a finite collection $\{U_i := (a_i, \infty)\}_{i=1}^n$, we have $\bigcap_{i=1}^n (a_i, \infty) = (b_0, \infty)$, where $b_0 = \max_{1 \leq i \leq n} \{a_i\}$. Again, we can ignore any $U_i = \mathbb{R}$, and if $U_i = \emptyset$ then the intersection is clearly empty.

Thus, $\mathcal{T}_{\rightarrow}$ is a topology on $\mathbb{R}$.

Q2. Compare (i.e., strictly fine, strictly coarse or incomparable) $\mathcal{T}_{\rightarrow}$ with the following.

i) The usual topology on $\mathbb{R}$.

Solution. Clearly any (a, ∞) is open in the usual topology, but a bounded open interval (a, b) is not open in $\mathcal{T}_{\rightarrow}$. Thus, $\mathcal{T}_{\rightarrow}$ is strictly coarser than the usual topology.

ii) The lower limit topology $\mathbb{R}_l$.

Solution. The lower limit topology is strictly finer than the usual topology, and hence, is strictly finer than $\mathcal{T}_{\rightarrow}$ as well.

Alternatively,

$$(a, \infty) = \bigcup_{n \geq 1} \left[a + \frac{1}{n}, a + n\right)$$

is clearly open in the lower limit topology. But $[0, 1)$ is not open in $\mathcal{T}_{\rightarrow}$.

iii) The upper limit topology $\mathbb{R}_u$.

Solution. The upper limit topology is strictly finer than the usual topology, and hence, is strictly finer than $\mathcal{T}_{\rightarrow}$ as well.

Alternatively,

$$(a, \infty) := \bigcup_{n \geq 1} (a, a + n]$$

is clearly open in the upper limit topology. But $(0, 1]$ is not open in $\mathcal{T}_{\rightarrow}$.

Q3. Determine (with justification) the closures of the following sets in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$.

i) $(0, \infty)$.

Solution. For any x, an open set containing x will be of the form (y, ∞) for some $y < x$, and hence,

$$(y, \infty) \cap (0, \infty) = (\max\{0, y\}, \infty) \neq \emptyset.$$

Thus, $\overline{(0, \infty)} = \mathbb{R}$.

ii) $(-\infty, 0)$.

Solution. Any open set containing 0 will be of the form $(-\epsilon, \infty)$ for some $\epsilon > 0$, and hence, $(-\epsilon, 0) \cap (-\infty, 0) = (-\epsilon, 0) \neq \emptyset$. For any $x > 0$, we have $(-\infty, 0) \cap (\frac{x}{2}, \infty) = \emptyset$. Thus, $\overline{(-\infty, 0)} = (-\infty, 0]$.

iii) $\{0\}$.

Solution. For any $x \leq 0$, an open set containing x is of the form (y, ∞) with $y < x \leq 0$, and hence, $0 \in (y, \infty)$. Thus, x is a closure point. So, $0 \in (-\infty, 0] \subset \overline{\{0\}}$. But by ii), we have $(-\infty, 0]$ is closed. Hence, closure being the smallest closed set containing $\{0\}$, we have $\overline{\{0\}} = (-\infty, 0]$.

iv) $A = \{1, 2, \dots\}$.

Solution. By iii), it follows that $\overline{\{n\}} = (-\infty, n]$. Now, $n \in A \Rightarrow \overline{\{n\}} \subset \bar{A}$. So,

$$\bar{A} \supset \bigcup_{n \geq 1} \overline{\{n\}} = \bigcup_{n \geq 1} (-\infty, n] = \mathbb{R}.$$

Thus, $\bar{A} = \mathbb{R}$.

v) $B = \{-1, -2, \dots\}$.

Solution. Again by iii), we have

$$\bar{B} \supset \overline{\{-1\}} = (-\infty, -1]$$

Also, $B \subset (-\infty, -1]$, which is closed by ii). Thus, $\bar{B} = (-\infty, -1]$.

Q4. Determine (with justification) whether $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ is T_0, T_1, or T_2.

Solution. We have $\overline{\{0\}} = (-\infty, 0]$, and hence the topology is not T_1 (and hence, not T_2). For any $x \neq y \in \mathbb{R}$, without loss of generality, assume $x < y$. Then, $x \notin (x, \infty)$ but $y \in (x, \infty)$. Thus, the topology is T_0.

Q5. Prove or give counter-example to the following statements.

i) If a sequence (x_n) converges to x in the usual topology, then $x_n \to x$ in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ as well.

Solution. Since $\mathcal{T}_{\rightarrow}$ is coarser than the usual topology, convergence in the usual topology implies convergence in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$.

ii) If a sequence (x_n) converges to x in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$, then $x_n \to x$ in the usual topology as well.

Solution. Consider the sequence $x_n = n$. Then, $\{x_n\}$ does not converge in the usual topology. But for any $x \in \mathbb{R}$, we have $(x - \epsilon, \infty)$ contains all but finitely many natural numbers. It follows that x_n converges to any point in $\mathbb{R}$ in the topology $\mathcal{T}_{\rightarrow}$.

Q6. Given a T_1-space $(X, \mathcal{T})$ (with at least two points), prove that any continuous map $f : (\mathbb{R}, \mathcal{T}_{\rightarrow}) \rightarrow (X, \mathcal{T})$ is constant. Give an example of a space $(Y, \mathcal{S})$ with $Y = \{0, 1\}$, and a nonconstant continuous map $f : (\mathbb{R}, \mathcal{T}_{\rightarrow}) \rightarrow (Y, \mathcal{S})$.

Solution. Consider a continuous map $f : (\mathbb{R}, \mathcal{T}_{\rightarrow}) \rightarrow (X, \mathcal{T})$, where X is T_1. If possible, suppose f is nonconstant. Then, we have some $a \neq b \in \mathbb{R}$ such that $x = f(a) \neq f(b) = y \in X$. Now, X is T_1 and hence, $\{x\}$ and $\{y\}$ are closed. Then, we have $a \in f^{-1}(x)$ and $b \in f^{-1}(y)$, two closed sets. Since these closed sets are not $\mathbb{R}$ (as f is nonconstant), we must have

$$f^{-1}(x) = (-\infty, a'], \quad f^{-1}(y) = (-\infty, b'],$$

for some $a \leq a', b \leq b'$. But then the closed set intersects, contradicting $x \neq y$. Hence, f must be constant.

Consider the space $Y = \{0, 1\}$ with the topology

$$\mathcal{S} = \{\emptyset, \{1\}, \{0, 1\}\}.$$

Define the map

$$f : \mathbb{R} \rightarrow Y$$
$$x \mapsto \begin{cases} 1, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Alternatively, consider the indiscrete topology on Y. Then, any map into Y (from any space) is always continuous. In particular, we can take any nonconstant map $\mathbb{R} \rightarrow Y$.

Q7. Consider the equivalence relation : $a \sim b$ if and only if $a - b \in \mathbb{Z}$. Show that the induced quotient space is an indiscrete space.

Solution. Observe that $\overline{\mathbb{Z}} = \mathbb{R}$ in the topology $(\mathbb{R}, \mathcal{T}_{\rightarrow})$. Now, consider the quotient map $q : \mathbb{R} \rightarrow \mathbb{R}/_{\sim}$. A set $C \subset \mathbb{R}/_{\sim}$ is closed if and only if $q^{-1}(C)$ is closed in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$. If possible, suppose $\emptyset \subsetneq C \subsetneq \mathbb{R}/_{\sim}$ is a closed set. Then $q^{-1}(C)$ is closed, and $\emptyset \neq q^{-1}(C) \neq \mathbb{R}$. Hence, we must have

$$q^{-1}(C) = (-\infty, a]$$

for some a. But then, there is some integer $n_0 \in q^{-1}(C)$. This implies,

$$\begin{aligned} n_0 \in q^{-1}(C) \Rightarrow q(n_0) \in C \Rightarrow \mathbb{Z} = q^{-1}\left(q(n_0)\right) \subset q^{-1}(C) \\ \Rightarrow \mathbb{R} = \overline{\mathbb{Z}} \subset \overline{q^{-1}(C)} = q^{-1}(C) \Rightarrow q^{-1}(C) = \mathbb{R}, \end{aligned}$$

which is a contradiction. Since C was arbitrary closed set, we have $\mathbb{R}/_{\sim}$ is indiscrete.

Q8. Consider the equivalence relation : $a \sim b$ if and only if either

$$a, b \in \mathbb{R} \setminus \mathbb{Z}, \text{ and } a = b, \qquad \text{or,} \qquad a, b \in \mathbb{Z}.$$

Show that the induced quotient space is an indiscrete space.

Solution. Again, consider some closed set $\emptyset \subsetneq C \subsetneq \mathbb{R}/_{\sim}$. Then, we have $q^{-1}(C) = (-\infty, a]$ for some a. But then again, there is some integer $n_0 \in q^{-1}(C)$. We get $\mathbb{Z} = q^{-1}(q(n_0)) \subset q^{-1}(C) \Rightarrow \mathbb{R} = \overline{\mathbb{Z}} \subset \overline{q^{-1}(C)} = q^{-1}(C) \Rightarrow q^{-1}(C) = \mathbb{R}$, a contradiction. Thus, the quotient topology $\mathbb{R}/_{\sim}$ is an indiscrete space.