Quiz 1

11th September, 2025

Time: 2 hrs **Marks:** ____/20

On the real line $\mathbb{R}$, consider the collection of subsets

$$\mathcal{T}_{\rightarrow} := \{\emptyset, \mathbb{R}\} \bigcup \{(a, \infty) \mid a \in \mathbb{R}\}.$$

Attempt any question. You can get **maximum 20**.

Q1. Show that $\mathcal{T}_{\rightarrow}$ is a topology on $\mathbb{R}$. $[2]$

Q2. Compare (i.e., strictly fine, strictly coarse or incomparable) $\mathcal{T}_{\rightarrow}$ with the following. $[1 \times 3 = 3]$

i) The usual topology on $\mathbb{R}$.

ii) The lower limit topology $\mathbb{R}_l$.

iii) The upper limit topology $\mathbb{R}_u$.

Q3. Determine (with justification) the closures of the following sets in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$. $[1 \times 5 = 5]$

i) $(0, \infty)$.

ii) $(-\infty, 0)$.

iii) $\{0\}$.

iv) $A = \{1, 2, \dots\}$.

v) $B = \{-1, -2, \dots\}$.

Q4. Determine (with justification) whether $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ is T_0, T_1, or T_2. $[1 \times 3 = 3]$

Q5. Prove or give counter-example to the following statements. $[1 \times 2 = 2]$

i) If a sequence (x_n) converges to x in the usual topology, then $x_n \to x$ in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ as well.

ii) If a sequence (x_n) converges to x in $(\mathbb{R}, \mathcal{T}_{\rightarrow})$, then $x_n \to x$ in the usual topology as well.

Q6. Given a T_1-space $(X, \mathcal{T})$ (with at least two points), prove that any continuous map $f : (\mathbb{R}, \mathcal{T}_{\rightarrow}) \to (X, \mathcal{T})$ is constant. Give an example of a space $(Y, \mathcal{S})$ with $Y = \{0, 1\}$, and a nonconstant continuous map $f : (\mathbb{R}, \mathcal{T}_{\rightarrow}) \to (Y, \mathcal{S})$. $[2 + 1 = 3]$

Q7. Consider the equivalence relation : $a \sim b$ if and only if $a - b \in \mathbb{Z}$. Show that the induced quotient space is an indiscrete space. $[4]$

Q8. Consider the equivalence relation : $a \sim b$ if and only if either

$$a, b \in \mathbb{R} \setminus \mathbb{Z}, \text{ and } a = b, \qquad \text{or,} \qquad a, b \in \mathbb{Z}.$$

Show that the induced quotient space is an indiscrete space. $[4]$

Definitions

1. The *lower limit topology* on $\mathbb{R}$ is generated by the basis $\{[a, b) \mid a, b \in \mathbb{R}\}$.

2. The *upper limit topology* on $\mathbb{R}$ is generated by the basis $\{(a, b] \mid a, b \in \mathbb{R}\}$.

3. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$ (and $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$). If $\mathcal{T}_1 \not\subset \mathcal{T}_2$ and $\mathcal{T}_2 \not\subset \mathcal{T}_1$, then they are *incomparable*.

4. Given a space X, we say

   (a) X is T_0 if given any two points $x \neq y \in X$, there exists some open set $U \subset X$ such that either $x \in U, y \notin U$ or $x \notin U, y \in U$ (i.e., U contains exactly one of $\{x, y\}$).

   (b) X is T_1 if any singleton subset of X is closed.

   (c) X is T_2 if given any two $x \neq y \in x$, there are open neighborhoods $x \in U, y \in V$ such that $U \cap V = \emptyset$.

5. A sequence $\{x_n\}$ in a topological space X converges to some point $x \in X$ if for any open neighborhood $x \in U \subset X$, we have some number $N = N_U \geq 1$, such that $x_n \in U$ for all $n \geq N$.

6. Given any equivalence relation $\sim$ on a space $(X, \mathcal{T})$, the induced quotient space is

$$X/_{\sim} := \{[x] := \{y \in X \mid x \sim y\} \mid x \in X\},$$

   with the quotient topology

$$\mathcal{T}_q := \left\{U \subset X/_{\sim} \,\middle|\, q^{-1}(U) \in \mathcal{T}\right\},$$

   where $q : X \to X/_{\sim}$, given by $q(x) = [x]$, is the quotient map.