Mid-semester Examination (Solutions)

Course : Topology (KSM1C03)

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Q1. (Furstenberg) Consider the integers $\mathbb{Z}$. For $a, b \in \mathbb{Z}$ with $a \neq 0$, denote the set

$$P(a,b) \coloneqq a\mathbb{Z} + b = \{an + b \mid n \in \mathbb{Z}\} = \{b, b \pm a, b \pm 2a, \dots\} \subset \mathbb{Z}.$$

a) Show that $\mathcal{B} \coloneqq \{P(a,b) \mid a, b \in \mathbb{Z},\ a \neq 0\}$ is a basis for a topology, say, $\mathcal{T}$ on $\mathbb{Z}$.

Solution: Clearly $P(1,0) = 1.\mathbb{Z} + 0 = \mathbb{Z}$. Suppose $x \in P(a,b) \cap P(c,d)$ for some $a, c \neq 0$. Then, we can write

$$x = ma + b = nc + d,$$

for some $m, n \in \mathbb{Z}$. Consider

$$l = \operatorname{lcm}(a,c) \neq 0.$$

Then, we have

$$l = a_1 a = c_1 c,$$

for some $a_1, c_1 \in \mathbb{Z}$. We claim that

$$x \in P(l, x) \subset P(a,b) \cap P(c,d).$$

Say, $y = x + kl \in P(l,x)$ for some $k \in \mathbb{Z}$. Then,

$$y = x + kl = ma + b + ka_1 a = b + (m + ka_1)a \in P(a,b),$$

and

$$y = x + kl = nc + d + kc_1 c = d + (n + kc_1)c \in P(c,d).$$

Hence, $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

b) Prove that any basic open set $P(a,b) \in \mathcal{B}$ is also closed in $(\mathbb{Z}, \mathcal{T})$.

Solution: Note that $P(a,b) = P(-a,b)$, and so, we can assume $a \geq 1$. We claim that

$$\mathbb{Z} \setminus P(a,b) = \bigcup_{j=1}^{a-1} P(a, b+j).$$

Note that for $a = 1$, the right-hand side is empty, and we clearly have $P(1,b) = \mathbb{Z} + b = \mathbb{Z}$, which is closed. If possible, suppose for some $1 \leq j \leq a-1$, we have

$$x = na + b + j = ma + b \Rightarrow na + j = ma \Rightarrow j = (m-n)a.$$

This is impossible. Thus,

$$P(a,b) \cap \left(\bigcup_{j=1}^{a-1} P(a,b+j) \right) = \emptyset \Rightarrow \bigcup_{j=1}^{a-1} P(a,b+j) \subset \mathbb{Z} \setminus P(a,b).$$

Next, suppose $x \notin P(a,b)$. By the Euclidean algorithm, we have $x - b = na + r$ for some $0 \le r < a$. But then,

$$x = na + b + r \in P(a, b+r).$$

Thus, $\mathbb{Z} \setminus P(a,b) = \bigcup_{j=1}^{a-1} P(a,b+j)$.

Alternative solution: Without loss of generality, one can assume that $a > 0$ and $0 \le b < a-1$. Next, one can easily observe

$$\mathbb{Z} = a\mathbb{Z} \sqcup (a\mathbb{Z}+1) \sqcup \ldots \sqcup (a\mathbb{Z}+a-1).$$

Then, we have

$$\mathbb{Z} \setminus P(a,b) = a\mathbb{Z} \sqcup (a\mathbb{Z}+1) \sqcup \ldots (a\mathbb{Z}+b-1) \sqcup (a\mathbb{Z}+b+1) \sqcup (a\mathbb{Z}+a-1),$$

which is a union of (basic) open sets. Thus, $P(a,b)$ is closed.

c) Justify that one can write : $\mathbb{Z} \setminus \{1,-1\} = \bigcup_{p \text{ is a prime}} P(p,0)$.

<u>Solution:</u> For any $n \neq \pm 1$, there exists some prime number p such that $n = n_1 p$. Then, $n \in P(p,0)$. Clearly, no prime number divides ± 1, and so, $\pm 1 \notin P(p,0)$ for any prime p. Thus, $\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{p \text{ is a prime}} P(p,0)$.

d) Prove that there are infinitely many prime numbers.

<u>Solution:</u> Suppose, there are finitely many prime numbers, say $\{p_1, \ldots, p_k\}$. Now,

$$\mathbb{Z} \setminus \{\pm 1\} = \bigcup_{i=1}^{k} P(p_i, 0)$$

is then a finite union of *closed sets*, and hence, $\{\pm 1\}$ is open. But any basic open set is infinite. Thus, $\{\pm 1\}$ cannot be open, a contradiction. Hence, there are infinitely many prime numbers.

Q2. Suppose X is an infinite set, equipped with the cofinite topology. Prove the following.

a) X is compact.

<u>Solution:</u> Suppose $X = \bigcup_{\alpha \in I} U_\alpha$ is an open cover. Fix some $x \in X$, and then $x \in U_{\alpha_0}$ for some $\alpha_0 \in I$. Now, $U = X \setminus \{x_1, \ldots, x_k\}$. Then, we have $x_i \in U_{\alpha_i}$ for some $\alpha_i \in I$. Clearly, $X = \bigcup_{i=0}^{k} U_{\alpha_i}$ is a finite sub-cover.

b) If $\{x_n\}$ is a sequence in X such that no point is repeated infinitely many times, then x_n converges to every point of X.

Solution: Let $x \in X$ be fixed. Say $x \in U \subset X$ is some open neighborhood. Then, $X \setminus U = \{y_1, \ldots, y_k\}$. Now, in the sequence $\{x_n\}$, none of the values are repeated infinitely many times. In particular, the set

$$F_j = \{x_n \mid x_n = y_j\}$$

is finite for all $1 \leq j \leq k$. Hence, there exists some $N \geq 1$ such that $x_n \notin \{y_1, \ldots, y_k\}$ for all $n \geq N$. But then $x_n \in U$ for all $n \geq N$. Thus, $x_n \to x$.

c) If $\{x_n\}$ is a sequence in X such that exactly one point, say y, is repeated infinitely many times, then x_n converges to only y, and no other point of X.

Solution: Suppose, the point y is repeated infinitely many times in x_n, and no other point is repeated infinitely many times. Say, $y \in U$ is an open neighborhood. Then, $X \setminus U = \{y_1, \ldots, y_k\}$. Clearly, $y \notin \{y_1, \ldots, y_k\}$. Again from the hypothesis, there exists some $N \geq 1$ such that $x_n \notin \{y_1, \ldots, y_k\}$ for all $n \geq N$. Thus, $x_n \in U$ for all $n \geq N$. As U is an arbitrary open neighborhood, we have $x_n \to y$.

Now, if possible, suppose $x_n \to z \neq y$. Consider $y \in V := X \setminus \{y\}$. Then, there exists some $N \geq 1$ such that $x_n \in V$ for all $n \geq N$. But this contradicts that $x_n = y$ for infinitely many values of n. Hence, $x_n \not\to z \neq y$.

Now, suppose $\{x_n\}$ is some arbitrary sequence in X which converges to some x. Show that the sequence must be either of type $b)$ or of type $c)$.

Solution: Suppose $x_n \to x$. If $\{x_n\}$ is of type b) or of type c), we are done. If not, then there are at least two distinct values, say, $y_1 \neq y_2$ is repeated infinitely many times in $\{x_n\}$. Without loss of generality, assume $x \neq y_2$. Then, we have an open neighborhood $x \in U = X \setminus \{y_2\}$. Since $x_n \to x$, there exists some $N \geq 1$ such that $x_n \in U$ for all $n \geq N$. But this contradicts that $x_n = y_2$ for infinitely many values of n. Thus, $x_n \not\to x$. This proves the claim.

Q3. Let X be a space.

a) Given a locally finite collection $\{F_\alpha\}_{\alpha \in I}$ of subsets of X, show that $\overline{\bigcup_{\alpha \in I} F_\alpha} = \bigcup_{\alpha \in I} \overline{F_\alpha}$.

Solution: For all α, we have

$$F_\alpha \subset \bigcup_\alpha F_\alpha \subset \overline{\bigcup_\alpha F_\alpha} \Rightarrow \overline{F_\alpha} \subset \overline{\bigcup_\alpha F_\alpha},$$

and hence, $\bigcup_\alpha \overline{F_\alpha} \subset \overline{\bigcup_\alpha F_\alpha}$. Conversely, suppose

$$x \in \overline{\bigcup_\alpha F_\alpha} \setminus \bigcup_{\alpha \in I} \overline{F_\alpha}.$$

Now, since $\{F_\alpha\}$ is a locally finite collection, there exists an open neighborhood $x \in U \subset X$, such that U intersects only finitely many of $\{F_\alpha\}$. Thus, there is a finite subset of indices $J \subset I$ (possibly empty!) such that

$$U \cap F_\alpha = \emptyset, \qquad \alpha \in I \setminus J.$$

Now,

$$V = U \setminus \bigcup_{\alpha \in J} \overline{F_\alpha}$$

is an open set. Also, $x \in V$, since $x \notin \bigcup_{\alpha \in I} \overline{F_\alpha}$. But clearly, $V \cap F_\alpha = \emptyset$ for all $\alpha \in I$, and thus,

$$V \cap \left(\bigcup_\alpha F_\alpha \right) = \emptyset.$$

This contradicts $x \notin \overline{\bigcup_{\alpha \in I} F_\alpha}$. Hence, we have $\overline{\bigcup_{\alpha \in I} F_\alpha} = \bigcup_{\alpha \in I} \overline{F_\alpha}$.

b) Suppose $\mathcal{C} = \{C_\alpha\}_{\alpha \in \mathcal{I}}$ is a locally finite collection of closed subsets of X, so that $X = \bigcup_{\alpha \in \mathcal{I}} C_\alpha$. For some space Y, let $f_\alpha : C_\alpha \to Y$ be a collection of continuous functions such that $f_\alpha(x) = f_\beta(x)$ for any $x \in C_\alpha \cap C_\beta$. Then, prove that there exists a unique continuous function $h : X \to Y$ such that $h(x) = f_\alpha(x)$ whenever $x \in C_\alpha$.

Solution: Define $h : X \to Z$ by

$$h(x) = f_\alpha(x), \quad \text{if } x \in C_\alpha.$$

Since for any $x \in C_\alpha \cap C_\beta$ we have $f_\alpha(x) = f_\beta(x)$, it follows that h is well-defined. Since $X = \bigcup_{\alpha \in I} C_\alpha$, clearly h is the unique map satisfying $h|_{C_\alpha} = f_\alpha$. To show h is continuous, let $F \subset Y$ be an arbitrary closed set. Now,

$$h^{-1}(F) = \bigcup_{\alpha \in I} f_\alpha^{-1}(F).$$

Since $f_\alpha : C_\alpha \to Y$ is continuous, we have $f_\alpha^{-1}(F)$ is closed in C_α. Since C_α is closed, we have $f_\alpha^{-1}(F)$ is closed in X. Finally, since $\{C_\alpha\}$ is a locally finite family, it follows that $\{f_\alpha^{-1}(F)\}_{\alpha \in I}$ is also locally finite. Hence,

$$\overline{f^{-1}(F)} = \overline{\bigcup_{\alpha \in I} f_\alpha^{-1}(F)} = \bigcup_{\alpha \in I} \overline{f_\alpha^{-1}(F)} = \bigcup_{\alpha \in I} f_\alpha^{-1}(F) = f^{-1}(F).$$

Thus, $f^{-1}(F)$ is closed. Hence, f is continuous.

c) Give an example of an infinite collection of closed sets, where the above pasting argument fails.

Solution: Consider the functions,

$$f_n : \left[-1, -\frac{1}{n}\right] \to \mathbb{R} \qquad\qquad f_0 : [0, 1] \to \mathbb{R}$$
$$x \mapsto 0, \qquad\qquad x \mapsto 1.$$

Then, $[0,1] = \bigcup_{n\geq 1}[-1, -\frac{1}{n}] \cup [0,1]$. Also, these functions pathc nicely to give the function

$$h : [-1,1] \to \mathbb{R}$$
$$x \mapsto \begin{cases} 0, & x < 0 \\ 1, & x \geq 1. \end{cases}$$

Clearly, h is not continuous.

Q4. Let X be a compact, T_2 space. Consider the identification space $Z := \frac{X\times[0,1]}{X\times\{0,1\}}$, and the one-point compactification $\hat{Y}$ of $Y := X \times (0,1)$. Prove the following. $2+3+5=10$

(a) Z is compact.

Solution: Since Z is a quotient space of a compact space $X \times [0,1]$, we have Z is compact.

(b) Y is locally compact, T_2.

Solution: Clearly, Y is T_2, being the product of T_2-spaces. Consider a basic open set $U \times V \subset Y$, and some point $(x,t) \in U \times V$. Since both X and $(0,1)$ are locally compact, we have compact neighborhoods A, B such that $x \in \mathring{A} \subset A \subset U$ and $y \in \mathring{B} \subset B \subset V$. Then, $(x,y) \in \mathring{A} \times \mathring{B} \subset A \times B \subset U \times V$. Clearly, $A \times B$ is compact, and $(x,y) \in \mathring{A} \times \mathring{B} \subset \operatorname{int}(A \times B)$. Thus, Y is locally compact.

(c) Z is homeomorphic to $\hat{Y}$.

Solution: Consider the map $f : X \times [0,1] \to \hat{Y}$ defined by

$$f(x,t) = \begin{cases} (x,t), & \text{if } 0 < t < 1, \\ \infty, & \text{if } t = 0, \text{ or } t = 1. \end{cases}$$

Let us check that f is continuous. For any open set $U \subset Y \subset \hat{Y}$, we have $f^{-1}(U) = U \subset X \times (0,1) \subset X \times [0,1]$, as $f|_{X\times(0,1)}$ is the identity map. As $X \times (0,1)$ is open, we have $f^{-1}(U)$ is open. Next, consider some open neighborhood V of ∞. Then $V = \{\infty\} \cup (Y \setminus C)$, where $C \subset Y$ is a compact set (which is also closed, as the space is T_2). Now, $f^{-1}(C) = C$ is again compact in $X \times (0,1)$ and hence in $X \times [0,1]$. Then,

$$f^{-1}(V) = f^{-1}(\infty) \cup f^{-1}(Y \setminus C) = X \times \{0,1\} \cup (Y \setminus C) = X \times [0,1] \setminus C.$$

As C is closed, we have $f^{-1}(V)$ is open. Thus, f is continuous.

Now, $f|_{X\times\{0,1\}}$ is constant, and hence, we have an induced map $\tilde{f} : Z = \frac{X\times[0,1]}{X\times\{0,1\}} \to \hat{Y}$, which is continuous by the property of quotient topology. Clearly, $\hat{f}$ is a bijection. Finally, Z is compact, and $\hat{Y}$ is T_2 as it is the one-point compactification of a locally compact, T_2 space. Hence, $\tilde{f}$ is an open map. But then $\tilde{f} : Z \to \hat{Y}$ is a homeomorphism.

Q5. Prove (or disprove) the following.

a) For any subspace $A \subset X$, we have $X \setminus \overline{X \setminus A} = \operatorname{int}(A)$.

Solution: Since $\overline{X \setminus A}$ is a closed set, we have $X \setminus \overline{X \setminus A}$ is open. Hence,

$$X \setminus A \subset \overline{X \setminus A} \Rightarrow X \setminus \overline{X \setminus A} \subset X \setminus (X \setminus A) = A \Rightarrow X \setminus \overline{X \setminus A} \subset \operatorname{int}(A).$$

Also, for any $x \in \operatorname{int}(A) \subset A$, we have an open neighborhood $U = \operatorname{int}(A)$ such that $U \cap (X \setminus A) = \emptyset$. Thus, $x \notin \overline{X \setminus A} \Rightarrow x \in X \setminus \overline{X \setminus A}$. Hence, we have $X \setminus \overline{X \setminus A} = \operatorname{int}(U)$.

b) For any subspace $A \subset X$, we have $\operatorname{int}(A) = \operatorname{int}\left(\overline{\operatorname{int}(A)}\right)$.

Solution: Consider $A = [0, 1) \cup (1, 2] \subset \mathbb{R}$. Then, $\operatorname{int}(A) = (0, 1) \cup (1, 2)$. On the other hand,

$$\operatorname{int}\left(\overline{\operatorname{int}(A)}\right) = \operatorname{int}\left(\overline{(0,1) \cup (1,2)}\right) = \operatorname{int}([0, 2]) = (0, 2).$$

Thus, the statement is not always true.

c) For any subspace $A \subset X$, we have $\overline{\operatorname{int}(A)} = \overline{\operatorname{int}\left(\overline{\operatorname{int}(A)}\right)}$.

Solution: We have

$$\operatorname{int}(A) \subset \overline{\operatorname{int}(A)} \Rightarrow \operatorname{int}(A) \subset \operatorname{int}\left(\overline{\operatorname{int}(A)}\right),$$

as the interior is the largest open set contained in a set. Taking closure, we have

$$\overline{\operatorname{int}(A)} \subset \overline{\operatorname{int}\left(\overline{\operatorname{int}(A)}\right)}.$$

On the other hand,

$$\operatorname{int}\left(\overline{\operatorname{int}(A)}\right) \subset \overline{\operatorname{int}(A)} \Rightarrow \overline{\operatorname{int}\left(\overline{\operatorname{int}(A)}\right)} \subset \overline{\operatorname{int}(A)}.$$

Hence, we have the equality $\overline{\operatorname{int}(A)} = \overline{\operatorname{int}\left(\overline{\operatorname{int}(A)}\right)}$.

d) A compact space is first countable at least at one point.

Solution: Consider the cofinite topology on $\mathbb{R}$. It is compact since any cofinite topology is compact. For a point $x \in \mathbb{R}$, if possible, let $\{U_n\}$ be a countable neighborhood basis. We have $F_n = \mathbb{R} \setminus U_n$ is finite, and hence, $F = \bigcup_{n=1}^{\infty} F_n$ is at most countably infinite. We have some $y \in \mathbb{R} \setminus (F \cup \{x\})$. Then, $V = \mathbb{R} \setminus \{y\}$ is an open neighborhood of x. Clearly, for any n, we have

$$U_n \subset V \Rightarrow \mathbb{R} \setminus F_n \subset \mathbb{R} \setminus \{y\} \Rightarrow y \in F_n,$$

a contradiction. Thus, $\{U_n\}$ is not a neighborhood basis at x. As $x \in \mathbb{R}$ is arbitrary, we see that $\mathbb{R}$ with cofinite topology is not first countable at any point.

Q6. Show that a function $f : X \to Y$ is continuous if and only if for any subset $A \subset X$, we have $f\left(\bar{A}\right) \subset \overline{f(A)}$.

Solution: Suppose f is continuous. Now, for any $A \subset Y$, we have $\overline{f(A)} \subset Y$ is closed. Then, $f^{-1}\left(\overline{f(A)}\right)$ is closed in X. We have,

$$f(A) \subset \overline{f(A)} \Rightarrow A \subset f^{-1}(f(A)) \subset f^{-1}\left(\overline{f(A)}\right) \Rightarrow \overline{A} \subset f^{-1}\left(\overline{f(A)}\right) \Rightarrow f(\overline{A}) \subset \overline{f(A)}.$$

Conversely, suppose $f(\overline{A}) \subset \overline{f(A)}$ for any $A \subset X$. For any $C \subset Y$ closed, we then have

$$f\left(\overline{f^{-1}(C)}\right) \subset \overline{f\left(f^{-1}(C)\right)} = \overline{C} = C \Rightarrow \overline{f^{-1}(C)} \subset f^{-1}(C).$$

Thus, $f^{-1}(C) = \overline{f^{-1}(C)}$, i.e, $f^{-1}(C)$ is closed. Hence, f is continuous.

Q7. Suppose X is a topological space. Show that the topology on X is indiscrete if and only if given any space Y, any function $f : Y \to X$ is continuous.

Solution: Suppose X is indiscrete. Then the only open sets are $\emptyset$ and X. Now, for any function $f : Y \to X$, we have $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = Y$ are open in Y. Thus, f is continuous.

Conversely, suppose for any space Y and function $f : Y \to X$ we have f is continuous. Consider Y to be X equipped with the indiscrete topology. Then, $\mathrm{Id} : Y \to X$ is a continuous map. Now, for any open set $\emptyset \neq U \subsetneq X$, we have $\mathrm{Id}^{-1}(U) = U$ is open in Y, which contradicts that Y is indiscrete. Thus, there are no nontrivial open sets in X. In other words, X is indiscrete.

Q8. Show that the product of a Lindelöf space X and a compact space Y is again Lindelöf.

Solution: Suppose X is Lindelöf and Y is compact. Consider an open cover $\{U_\alpha\}$ of $X \times Y$. For each $x \in X$, we have $\{x\} \times Y$ is a compact set in $X \times Y$. Hence, there is a finite sub-cover

$$\{x\} \times Y \subset \bigcup_{\alpha \in J_x} U_\alpha,$$

where J_x is a finite indexing set. By the tube lemma, there exists $x \in O_x \underset{\text{open}}{\subset} X$, such that

$$\{x\} \times Y \subset O_x \times Y \subset \bigcup_{\alpha \in J_x} U_\alpha.$$

We now have a cover $X = \bigcup_{x \in X} O_x$. Since X is Lindelöf, there is a countable sub-cover, say, $X = \bigcup_{i=1}^{\infty} O_{x_i}$. Hence, we have

$$X \times Y = \bigcup_{i=1}^{\infty} O_{x_i} \times Y \subset \bigcup_{i=1}^{\infty} \bigcup_{\alpha \in J_{x_i}} U_\alpha.$$

Since a countable union of finite sets is again countable, we have a countable sub-cover of $X \times Y$. Thus, $X \times Y$ is Lindelöf.

Q9. Let X be a second countable space. Show that there exists a countable subset $A \subset X$, such that $X = \bar{A}$.

Solution: Let $\mathcal{B} = \{B_i\}$ be a countable basis of X. For each i, choose some $x_i \in B_i$. Then, $A = \{x_i\}$ is clearly a countable set. We claim that $X = \bar{A}$. For any $x \in X$, consider an open neighborhood $x \in U$. Then, there exists some i_0 so that $x \in B_{i_0} \subset U$. Now, $x_{i_0} \in B_{i_0} \subset U \Rightarrow U \cap A \neq \emptyset$. Thus, $x \in \bar{A}$. Since $x \in X$ is arbitrary, we have $X = \bar{A}$.

Q10. Let X, Y be given spaces. For any $K \subset X$, and $U \subset Y$, consider the collection of continuous maps

$$W(K,U) := \{f : X \to Y \mid f \text{ is continuous, } f(K) \subset U\}.$$

Next, consider the collection

$$\mathcal{S} := \{W(K,U) \mid K \subset X \text{ is compact, } U \subset Y \text{ is open}\}.$$

The topology on

$$Y^X := \text{Map}(X,Y) = \{f : X \to Y \text{ continuous}\}$$

generated by $\mathcal{S}$ as a sub-basis, is called the *compact-open* topology.

a) Suppose X is locally compact. Show that the evaluation map

$$\begin{aligned} ev : Y^X \times X &\longrightarrow Y \\ (f,x) &\longmapsto f(x) \end{aligned}$$

is continuous, where Y^X has the compact-open topology.

Solution: Say $U \subset Y$ is an open set. Let $(f,x) \in ev^{-1}(U)$. Then, we have

$$ev(f,x) = f(x) \in U \Rightarrow x \in f^{-1}(U).$$

Since X is locally compact, we have some compact set $K \subset X$ such that

$$x \in \text{int}(K) \subset K \subset f^{-1}(U).$$

Consider the (sub-basic) open set $W(K,U) \subset Y^X$. By construction, $f \in W(K,U)$. Observe that for any $(g,y) \in W(K,U) \times K$, since $g(K) \subset U$, we have

$$ev(g,y) = g(y) \in U \Rightarrow (g,y) \in ev^{-1}(U).$$

Thus, we have an open set,

$$(x,f) \in W(K,U) \times \text{int}(K) \subset ev^{-1}(U).$$

This proves that ev is a continuous map.

b) For any map $f : X \times Y \to Z$, define the *adjoint map* as

$$\begin{aligned} f^\wedge : X &\longrightarrow Z^Y \\ x &\longmapsto (y \mapsto f(x,y)) . \end{aligned}$$

Assume Z^Y has the compact-open topology.

i) Show that if f is continuous, then $f^\wedge$ is continuous.

Solution: Consider a sub-basic open set $W(K,U) \subset Z^Y$, where $K \subset Y$ is compact, and $U \subset Z$ is open. Consider some

$$x \in (f^\wedge)^{-1} (W(K,U)) .$$

Then, for any $y \in K$, we have

$$f^\wedge(x)(y) = f(x,y) \in U.$$

In other words, we have $\{x\} \times K \subset f^{-1}(U)$. Since f is continuous, we have $f^{-1}(U)$ is open. Then, by the tube lemma, there exists some open neighborhood $x \in V \subset X$, such that $V \times K \subset f^{-1}(U)$. Now, for any $v \in V$ and $y \in K$, we have

$$(f^\wedge)(v)(y) = f(v,y) \in U.$$

Hence, $x \in V \subset (f^\wedge)^{-1} (W(K,U))$. Thus, $f^\wedge$ is continuous.

ii) Suppose Y is locally compact. Show that if $f^\wedge$ is continuous then f is continuous

Solution: Observe that we have a commutative diagram

$$X \times Y \xrightarrow{f^\wedge \times \mathrm{Id}_Y} Z^Y \times Y \xrightarrow{ev} Z, \qquad X \times Y \xrightarrow{f} Z$$

Indeed, for any $(x,y) \in X \times Y$, we have

$$ev \left((f^\wedge \times \mathrm{Id}_Y)(x,y) \right) = ev \left(y' \mapsto f(x,y'), y \right) = f(x,y).$$

Thus, we have the equation

$$ev \circ (f^\wedge \times \mathrm{Id}_Y) = f.$$

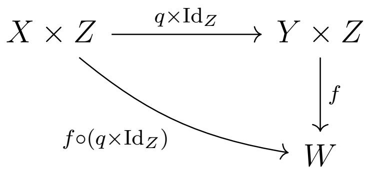
Since Y is locally compact, we see that $ev : Z^Y \times Y \to Z$ is continuous. Also, $f^\wedge \times \mathrm{Id}_Z$ is continuous, being the product of two continuous maps. Hence, f, being their composition, is also continuous.

c) (J.H.C. Whitehead) Suppose $q : X \to Y$ is a quotient map, and Z is locally compact. Show that the product

$$\begin{aligned} p := q \times \mathrm{Id}_Z : X \times Z &\longrightarrow Y \times Z \\ (x,z) &\longmapsto (q(x), z) \end{aligned}$$

is a quotient map.

Solution: Suppose $f : Y \times Z \to W$ is some arbitrary set map, such that, $f \circ p : X \times Z \to W$ is continuous.



Then, $(f \circ p)^\wedge : X \to W^Z$ is continuous. Now, we have the map $f^\wedge : Y \to W^Z$. Observe that for any $x \in X$, and $z \in Z$, we have

$$((f \circ p)^\wedge (x)) (z) = (f \circ p) (x, z) = f (q(x), z) = (f^\wedge (q(x))) (z).$$

Thus, we have

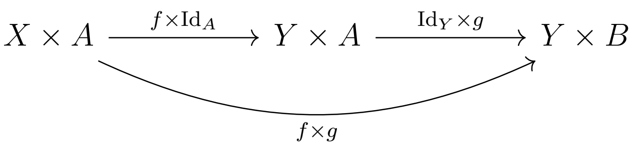
$$(f \circ p)^\wedge = f^\wedge \circ q.$$

Now, $f^\wedge \circ q$ is continuous, and q is given to be a quotient map. Hence, $f^\wedge : Y \to W^Z$ is continuous. Since Z is locally compact, this implies that f is continuous. But then by the universal property of the quotient map, we have $p = q \times \mathrm{Id}_Z$ is a quotient map.

d) Let $f : X \to Y$ and $g : A \to B$ be quotient maps, and Y, A be locally compact. Show that the product

$$\begin{aligned} q \coloneqq f \times g : X \times A &\longrightarrow Y \times B \\ (x, a) &\longmapsto (f(x), g(a)) \end{aligned}$$

Solution: It is easy to see that the diagram



commutes. Since A is locally compact, it follows that $f \times \mathrm{Id}_A$ is a quotient map. Similarly, since Y is locally compact, we have $\mathrm{Id}_Y \times g$ is a quotient map. Then, $f \times g$, being a composition of two quotient maps, is again a quotient map.