Mid-semester Examination

Course : Topology (KSM1C03)

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11th October, 2025

Time: 2:00 PM onwards **Total marks:** 90

Attempt any question. You can get maximum **70 marks**.

Q1. (Furstenberg) Consider the integers $\mathbb{Z}$. For $a, b \in \mathbb{Z}$ with $a \neq 0$, denote the set

$$P(a,b) \coloneqq a\mathbb{Z} + b = \{an + b \mid n \in \mathbb{Z}\} = \{b, b \pm a, b \pm 2a, \dots\} \subset \mathbb{Z}.$$

a) Show that $\mathcal{B} \coloneqq \{P(a,b) \mid a, b \in \mathbb{Z},\ a \neq 0\}$ is a basis for a topology, say, $\mathcal{T}$ on $\mathbb{Z}$.

b) Prove that any basic open set $P(a,b) \in \mathcal{B}$ is also closed in $(\mathbb{Z}, \mathcal{T})$.

c) Justify that one can write $\mathbb{Z} \setminus \{1, -1\} = \bigcup_{p \text{ is a prime}} P(p, 0)$.

d) Conclude that there are infinitely many prime numbers. $[3 + 3 + 2 + 2 = 10]$

Q2. Suppose X is an infinite set, equipped with the cofinite topology. Prove the following.

a) X is compact.

b) If $\{x_n\}$ is a sequence in X such that no point is repeated infinitely many times, then x_n converges to every point of X.

c) If $\{x_n\}$ is a sequence in X such that exactly one point, say y, is repeated infinitely many times, then x_n converges to only y, and no other point of X.

Now, suppose $\{x_n\}$ is some arbitrary sequence in X which converges to some x. Show that the sequence must be either of type $b)$ or of type $c)$. $[2 + 3 + 3 + 2 = 10]$

Q3. Let X be a space.

a) Given a locally finite collection $\{F_\alpha\}_{\alpha \in I}$ of subsets of X, show that $\overline{\bigcup_{\alpha \in I} F_\alpha} = \bigcup_{\alpha \in I} \overline{F_\alpha}$.

b) Suppose $\mathcal{C} = \{C_\alpha\}_{\alpha \in I}$ is a locally finite collection of closed subsets of X, so that $X = \bigcup_{\alpha \in I} C_\alpha$. For some space Y, let $f_\alpha : C_\alpha \to Y$ be a collection of continuous functions such that $f_\alpha(x) = f_\beta(x)$ for any $x \in C_\alpha \cap C_\beta$. Then prove that there exists a unique continuous function $h : X \to Y$ such that $h(x) = f_\alpha(x)$ whenever $x \in C_\alpha$.

c) Give an example of an infinite collection of closed sets, where the above pasting argument fails. $[4 + 4 + 2 = 10]$

Q4. Let X be a compact, T_2 space. Consider the identification space $Z \coloneqq \frac{X \times [0,1]}{X \times \{0,1\}}$, and the one-point compactification $\hat{Y}$ of $Y \coloneqq X \times (0,1)$. Prove the following. $[2 + (2 + 1) + 5 = 10]$

a) Z is compact.

b) Y is locally compact, T_2.

c) Z is homeomorphic to $\hat{Y}$.

Q5. Prove (or disprove) the following. $[2\frac{1}{2} \times 4 = 10]$

a) For any subspace $A \subset X$, we have $X \setminus \overline{X \setminus A} = \operatorname{int}(A)$.

b) For any subspace $A \subset X$, we have $\operatorname{int}(A) = \operatorname{int}\left(\overline{\operatorname{int}(A)}\right)$.

c) For any subspace $A \subset X$, we have $\overline{\operatorname{int}(A)} = \overline{\operatorname{int}\left(\overline{\operatorname{int}(A)}\right)}$.

d) A compact space is first countable at least at one point.

Q6. Show that a function $f : X \to Y$ is continuous if and only if for any subset $A \subset X$, we have $f\left(\bar{A}\right) \subset \overline{f(A)}$. $[5]$

Q7. Suppose X is a topological space. Show that the topology on X is indiscrete if and only if given any space Y, any function $f : Y \to X$ is continuous. $[5]$

Q8. Show that the product of a Lindelöf space X and a compact space Y is again Lindelöf. $[5]$

Q9. Let X be a second countable space. Show that there exists a countable subset $A \subset X$, such that $X = \bar{A}$. $[5]$

Q10. Let X, Y be given spaces. For any $K \subset X$, and $U \subset Y$, consider the collection of continuous maps

$$W(K,U) := \{f : X \to Y \mid f \text{ is continuous, } f(K) \subset U\}.$$

Next, consider the collection

$$\mathcal{S} := \{W(K,U) \mid K \subset X \text{ is compact, } U \subset Y \text{ is open}\}.$$

The topology on

$$Y^X := \text{Map}(X,Y) = \{f : X \to Y \mid f \text{ is continuous}\}$$

generated by $\mathcal{S}$ as a sub-basis, is called the *compact-open* topology.

a) Suppose X is locally compact. Show that the evaluation map

$$\begin{aligned} ev : Y^X \times X &\longrightarrow Y \\ (f,x) &\longmapsto f(x) \end{aligned}$$

is continuous, where Y^X has the compact-open topology.

b) For any map $f : X \times Y \to Z$, define the *adjoint map* as

$$\begin{aligned} f^\wedge : X &\longrightarrow Z^Y \\ x &\longmapsto (y \mapsto f(x,y)). \end{aligned}$$

Assume Z^Y has the compact-open topology.

i) Show that if f is continuous, then $f^\wedge$ is continuous. (Hint : Use the tube lemma!)

ii) Suppose Y is locally compact. Show that if $f^\wedge$ is continuous then f is continuous. (Hint : Write f in terms of $f^\wedge$ and a suitable evaluation map.)

c) (J.H.C. Whitehead) Suppose $q : X \to Y$ is a quotient map, and Z is locally compact. Show that the product

$$\begin{aligned} p := q \times \text{Id}_Z : X \times Z &\longrightarrow Y \times Z \\ (x,z) &\longmapsto (q(x), z) \end{aligned}$$

is a quotient map. (Hint : Use the universal property. Take a set map $f : Y \times Z \to W$ with $f \circ p$ continuous, and use the adjoint operation suitably.)

d) Let $f : X \to Y$ and $g : A \to B$ be quotient maps, and Y, A be locally compact. Show that the product

$$\begin{aligned} q := f \times g : X \times A &\longrightarrow Y \times B \\ (x,a) &\longmapsto (f(x), g(a)) \end{aligned}$$

is a quotient map. $[4 + (6 + 3) + 5 + 2 = 20]$

Definitions/Hints

- A collection $\mathcal{B} \subset \mathcal{P}(X)$ is a basis for a topology on X, if i) for any $x \in X$, there is some $B \in \mathcal{B}$, with $x \in B$, and ii) for any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.
- In the cofinite topology on a set X, a nonempty subset $U \subset X$ is open if and only if $X \setminus U$ is a finite set.
- The interior $\text{int}(A)$ is the largest open set contained in A, and the closure $\bar{A}$ is the smallest closed set containing A.
- A collection $\mathcal{A} = \{A_\alpha \subset X\}$ of subsets is called locally finite, if for any $x \in X$, there exists an open neighborhood $x \in U \subset X$, such that U intersects at most finitely many (possibly none) of A_α.
- X is locally compact if for any open set U and any $x \in U$, there exists a compact set C with $x \in \text{int}(C) \subset C \subset U$.
- A noncompact space X is locally compact, T_2 if and only if the one-point compactification $\hat{X}$ is T_2.
- A space is second countable if it admits a countable basis.
- A space X is first countable at a point $x \in X$ if there exists a countable collection of open neighborhoods $\{U_i\}$ of x, such that for any open neighborhood $x \in V$, there is some i_0 satisfying $x \in U_{i_0} \subset V$.
- A space X is called Lindelöf if given any open cover of X, there is a countable sub-cover.
- Given a subspace $A \subset X$, the identification space X/A is the quotient space induced by the equivalence relation: $x \sim y$ if and only if either (i) $x, y \notin A$ and $x = y$, or (ii) $x, y \in A$.
- Tube lemma : Let $x \in X$ and $C \subset Y$ be compact. If $\{x\} \times C \subset O \subset X \times Y$, where O is open, then there exists an open neighborhood $x \in U \subset X$ such that $\{x\} \times C \subset U \times C \subset O \subset X \times Y$.
- Universal property of the quotient map : A map $q : X \to Y$ is a quotient map if and only if for any function $f : Y \to W$, the map f is continuous precisely when $f \circ q$ is continuous.