End-semester Examination

Course : Topology (KSM1C03)

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6th December, 2025

Time: 2:00 PM onwards **Total marks:** 100

Attempt any **3** from **Q1** - **Q5**, any **3** from **Q6** - **Q10**, and **Q11** is mandatory. You can get maximum **70 marks**.

Q1. A topology $\mathcal{T}$ on X is said to be *minimally Hausdorff* if $(X, \mathcal{T})$ is a T_2-space, and given any strictly coarser topology $\mathcal{T}' \subsetneq \mathcal{T}$ on X, we have $(X, \mathcal{T}')$ is not T_2. Show that a compact, T_2 space is minimally Hausdorff.

Solution: Let $(X, \mathcal{T})$ be a compact T_2 space. If possible, suppose $\mathcal{T}' \subsetneq \mathcal{T}$ is a strictly coarser topology, which is again T_2. Then, there is a (nonempty) set $U \in \mathcal{T} \setminus \mathcal{T}'$. Now, $C = X \setminus U$ is closed in $(X, \mathcal{T})$, and hence, compact. But then C is compact in $(X, \mathcal{T}')$ as well, since $\mathcal{T}' \subset \mathcal{T}$. Now, $(X, \mathcal{T}')$ is T_2, and hence, C is closed in $(X, \mathcal{T}')$. This means, $U = X \setminus C$ is open in $(X, \mathcal{T}')$, a contradiction. Hence, $(X, \mathcal{T})$ is minimally T_2.

Q2. A space X is called *hereditarily connected* if every subspace of X is connected. Show that X is hereditarily connected if and only if the topology on X is a totally ordered set with respect to set inclusion (i.e., if and only if for any two open sets $U, V \subset X$ we have $U \subset V$ or $V \subset U$).

Solution: Suppose X is hereditarily connected. Let $U, V \subset X$ be open. Consider the symmetric difference

$$\Delta \coloneqq (U \setminus V) \cup (V \setminus U),$$

By hypothesis Δ is connected. But, $D \cap U = U \setminus V$ and $\Delta \cap V = V \setminus U$ are disjoint open sets of Δ, whose union is all of Δ. Hence, one of them must be empty. In other words, either $U \setminus V = \emptyset \Rightarrow U \subset V$, or $V \setminus U = \emptyset \Rightarrow V \subset U$. Hence, the topology on X is totally ordered.

Conversely, suppose the topology on X is totally ordered. Let $Y \subset X$ be any subset. If possible, let Y be disconnected. Then, there are open sets $U, V \subset X$ such that

$$(Y \cap U) \cap (Y \cap V) = \emptyset, \quad Y \subset U \cup V, \quad \emptyset \neq Y \cap U, Y \cap V \subsetneq Y.$$

Now, without loss of generality, we have $U \subset V$. But then, $Y \cap U \subset Y \cap V \Rightarrow Y \cap U = \emptyset$, a contradiction. Hence, Y must be connected. In other words, X is hereditarily connected.

Q3. Let X be a locally connected, separable space. Show that any open set $U \subset X$ can be written as a countable union of disjoint, open, connected sets.

Solution: Let $U \subset X$ be open. Since X is locally connected, connected components of U are open. Let us write $U = \bigsqcup_{\alpha \in \Lambda} U_\alpha$, where $U_\alpha \subset U$ are the connected components of U. Now, X is separable. Hence, there is a countable dense set, say, $Q \subset X$. Since each $U_\alpha \subset X$ is nonempty open, we can choose some $q_\alpha \in U_\alpha \cap Q$ for each $\alpha \in \Lambda$. Clearly, $\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \emptyset \Rightarrow q_\alpha \neq q_\beta$. Thus, we have an injective map $\Lambda \hookrightarrow Q$ given by $\alpha \mapsto q_\alpha$. Hence, Λ must be countable. Thus, any open set U can be written as a countable union of disjoint, open, connected sets.

Q4. Let X be a locally compact, T_2 space.

a) Show that X is $T_{3\frac{1}{2}}$.

Solution: Since X is locally compact T_2, the one-point compactification $\hat{X} = X \cup \{\infty\}$ is compact, T_2, and hence, T_4. In particular, $\hat{X}$ is $T_{3\frac{1}{2}}$. As $X \hookrightarrow \hat{X}$ is a subspace, we have X is $T_{3\frac{1}{2}}$.

b) If X is second countable, show that X is paracompact.

Solution: Suppose X is additionally second countable. Now, X is $T_{3\frac{1}{2}} \Rightarrow T_3$. Then, by the Urysohn metrization theorem, X is metrizable. But then X is paracompact as every metrizable space is paracompact.

Q5. Show that a perfectly normal, T_0-space is T_6.

Solution: Let X be a perfectly normal, T_0-space. We need to show that X is T_1. Let $x, y \in X$ be such that $x \neq y$. Since X is T_0, without loss of generality, there is an open set $U \subset X$ such that $x \in U$ but $y \notin U$. So, $y \in C := X \setminus U$, which is a closed set. Since X is perfectly normal, we have X is a G_δ-space. In particular, $C = \bigcap_{i=1}^{\infty} V_i$ for some open sets $V_i \subset X$. Since $x \notin C$, we have $x \in V_{i_0}$ for some i_0. Thus, we have two open sets U and V_{i_0}, each containing exactly one of x, y. Since x, y are arbitrary, we have X is T_1. But then X is T_6 by definition.

Q6. Let X be a T_2 space.

a) Suppose $f, g : Z \to X$ are continuous maps. Show that the set $E(f, g) := \{z \in Z \mid f(z) = g(z)\}$ is closed in Z.

Solution: Consider the map $h : Z \to X \times X$ given by $h(x) = (f(x), g(x))$, which is clearly continuous. Since X is T_2, we have the diagonal $\Delta = \{(x, x) \mid x \in X\} \subset X \times X$ is closed. Note that $E(f, g) = h^{-1}(\Delta)$. Hence, $E(f, g)$ is closed in Z.

b) Let $\iota : A \to X$, and $r : X \to A$ be continuous maps satisfying $r \circ \iota = \mathrm{Id}_A$. Show that ι is injective, and $\iota(A)$ is closed in X.

Solution: For any $a, b \in A$ we have $i(a) = i(b) \Rightarrow r(\iota(a)) = r(\iota(b)) \Rightarrow a = b$. Thus, ι is injective.

Consider two maps $f = \iota \circ r : X \to X$ and $g = \mathrm{Id}_X : X \to X$, which are clearly continuous. Note that for any $x \in X$, we have $f(x) = g(x) \Rightarrow x = \iota(r(x)) \in \iota(A)$. Also, for any $\iota(a) \in \iota(A)$ we have $f(\iota(a)) = \iota r \iota(a) = \iota(a) = g(\iota(a))$. Thus, $\iota(A) = E(f, g)$ is closed in X.

A subspace $A \subset X$ is called a *retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for any $a \in A$. Show that a retract of a T_2-space is a closed subset.

Solution: Consider the inclusion map $\iota : A \hookrightarrow X$, which is continuous as A is a subspace. Then, it follows that $A = \iota(A)$ is closed in X.

Q7. On $\mathbb{R}$, consider the particular point topology $\mathcal{T}_0$ with base 0, i.e,

$$\mathcal{T}_0 := \{\emptyset\} \cup \{A \subset \mathbb{R} \mid 0 \in A\}.$$

Denote $X = (\mathbb{R}, \mathcal{T}_0)$.

a) Which of the following properties does X have? Justify.

i) Lindelöf ii) Separable iii) Locally compact iv) Path connected.

Solution:

i) X is not Lindelöf.
The set $A = \mathbb{R} \setminus \{0\}$ is closed and discrete. As A is uncountable, A cannot be Lindelöf. Hence, X is not Lindelöf.

ii) X is separable.
Since any nonempty open set contains 0, it follows that the singleton $\{0\}$ is dense in X. Thus, X is separable.

iii) X is locally compact.

Let $U \subset X$ be open, and $x \in U$ be a point. We clearly have $0 \in C$. Consider the set $C = \{0, x\}$, which is open. Also, C being finite, is compact. Thus, $x \in C \subset U$ is a compact neighborhood of x. Hence, X is locally compact.

iv) X is path connected.

Let $x, y \in X$ be two points. Consider the map $f : [0,1] \to X$ defined by

$$f(t) = \begin{cases} x, & t = 0, \\ 0, & 0 < t < 1, \\ y, & t = 1. \end{cases}$$

We claim that f is continuous. For $t = 0$, consider $U = \{0, x\}$, which is open, and we have $f^{-1}(U) = [0, 1)$ (or $f^{-1}(U) = [0, 1]$ if $y = 0$). Thus, f is continuous at $t = 0$. By similar argument, f is continuous at $t = 1$. Now, suppose $0 < t < 1$. Consider $U = \{0\}$. Then, $f^{-1}(U) = (0, 1)$ (or, $[0, 1), (0, 1], [0, 1]$ depending on $x = 0, y = 0$ or $x = 0 = y$). Thus, f is continuous.

b) Explicitly describe all the open sets in the Alexandroff compactification $\hat{X} = X \cup \{\infty\}$.

Solution: By the construction, it follows that any open set in X is open in $\hat{X}$. Thus, all set $A \subset X$ such that $0 \in A$ is open in $\hat{X}$. Now, the open sets containing ∞ are of the form $\{\infty\} \cup X \setminus C$, where C is closed and compact in X. Thus, we need to classify all closed compact sets of X.

Note that the closed sets of C are precisely those that do not contain 0. But any such set is discrete. Hence, the only closed, compact sets are finite subsets of X that do not contain 0.

Hence, the topology on $\hat{X} = X \cup \{\infty\}$ is

$$\left\{\emptyset, \hat{X}\right\} \cup \{A \subset \mathbb{R} \mid 0 \in A\} \cup \{\{\infty\} \cup (\mathbb{R} \setminus F) \mid F \subset \mathbb{R} \text{ is finite, } 0 \notin F\}.$$

Q8. On $\mathbb{R}$, consider the following topology

$$\mathcal{T} := \{\emptyset, \mathbb{R}\} \cup \{S \mid S \subset \mathbb{R},\ 0 \notin S\} \cup \{\mathbb{R} \setminus C \mid C \subset \mathbb{R} \setminus \{0\} \text{ is countable}\}.$$

The space $X = (\mathbb{R}, \mathcal{T})$ is called the *fortissimo space* on $\mathbb{R}$.

a) Show that X is T_5.

Solution: Let us show that X is completely normal. Suppose $A, B \subset X$ are separated subsets, i.e, $A \cap \bar{B} = \emptyset = \bar{A} \cap B$. If $0 \notin A$, and $0 \notin B$, then clearly A, B are disjoint open sets. Thus, we have a separation of A, B by opens. Now, without loss of generality, suppose $0 \in A$. Then, $0 \notin \bar{B}$. As $0 \notin B$, we have B is open. We claim that B is closed as well. Indeed, for any $x \in \bar{B}$, we have $x \neq 0$. Then, for the open neighborhood $O = \{x\}$ of x to intersect B, we must have $x \in B$. Thus, $\bar{B} = B$. Then, $U = X \setminus B$ is an open set containing A, which is disjoint from the open set $V = B$. Thus, any two separated sets of X is separated by disjoint open sets. Consequently, X is completely normal.

Clearly, $\{0\}$ is a closed point, as $X \setminus \{0\} = \mathbb{R} \setminus \{0\}$ is open. Also, for any $x \neq 0$, we have $X \setminus \{x\} = \mathbb{R} \setminus \{x\}$ is a cocountable set containing 0, which is open. Thus, X is T_1. But then X is T_5.

b) Show that X is not T_6.

Solution: We show that the closed set $A = \{0\} \subset X$ is not G_δ. If possible, suppose $A = \bigcap U_i$ for open sets $U_i \subset X$. Since $0 \in U_i$, we have $U_i = \mathbb{R} \setminus C_i$ for countable subsets $C_i \subset \mathbb{R} \setminus \{0\}$ Then,

$$A = \bigcap U_i = \bigcap (\mathbb{R} \setminus C_i) = \mathbb{R} \setminus \bigcup C_i.$$

Since the countable union of countable sets is countable, we have $\bigcap C_i$ is countable. But then $\mathbb{R} \setminus \bigcup C_i$ is uncountable, which is a contradiction. Thus, A is not G_δ. Hence, X is not perfectly normal, and in particular, not T_6.

c) Show that X is Lindelöf, but not compact.

Solution: Let $\mathcal{U} = \{U_\alpha\}$ be an open cover X. Then, for some α_0 we have $0 \in U_{\alpha_0}$. As U_{α_0} is an open neighborhood of 0, we have $U_{\alpha_0} = \mathbb{R} \setminus C$ for some countable set $C = \{x_i\}_{i \geq 1} \subset \mathbb{R} \setminus \{0\}$. For each $i \geq 1$, we have some α_i such that $x_i \in U_{\alpha_i}$. Then, $\{U_{\alpha_i}\}_{i \geq 0}$ is a countable sub-cover of X. As $\mathcal{U}$ was arbitrary, we have X is Lindelöf.

On the other hand, consider the open sets

$$V_0 = \mathbb{R} \setminus \{1, 2, 3, \dots\}, \quad V_n = \{n\}, \; n \geq 1.$$

Clearly, $\mathcal{V} = \{V_i\}_{i=0}^\infty$ is an open cover of X, which does not admit any finite sub-cover.

d) Is X metrizable?

Solution: No, X is not metrizable. In fact, X is not first countable at 0. If we have a countable neighborhood basis $\{N_i\}$ at 0, then we have $\bigcap N_i = \mathbb{R} \setminus C$, for some countable set $C \subset \mathbb{R} \setminus \{0\}$. Choose any $0 \neq x \in \mathbb{R} \setminus C$. Then, $V = \mathbb{R} \setminus (C \cup \{x\})$ is an open neighborhood of 0. Clearly, none of N_i is contained in V. Thus, X is not a first countable space, and hence, not metrizable.

Q9. On $\mathbb{R}$, for each irrational x, fix a sequence $x_i \in \mathbb{Q}$ such that $x_i \to x$ (in the usual sense). Denote the set

$$U_n(x) = \{x\} \cup \{x_i \mid i > n\}, \quad x \in \mathbb{R} \setminus \mathbb{Q},\ n \geq 0.$$

Consider the collection of subsets

$$\mathcal{B} := \{\{q\} \mid q \in \mathbb{Q}\} \cup \{U_n(x) \mid x \in \mathbb{R} \setminus \mathbb{Q},\ n \geq 0\}.$$

Prove the following.

a) $\mathcal{B}$ is a basis for a topology, say, $\mathcal{T}$ on $\mathbb{R}$ (called the *rational sequence topology*).

Solution: Clearly, for each $x \in \mathbb{R}$, there is an element of $\mathcal{B}$ that contains x. Let $B_1, B_2 \in \mathcal{B}$, and suppose $x \in B_1 \cap B_2$. If x is a rational, then we can take $B_3 = \{x\}$ so that $x \in B_3 \subset B_1 \cap B_2$. If x is an irrational, then we must have $B_1 = U_n(x)$ and $B_2 = U_m(x)$ for some $m, n \geq 0$. Let $k = \max\{m, n\}$, and set $B_3 = U_k(x)$. Clearly, $x \in B_3 \subset B_1 \cap B_2$. Thus, $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

b) Each basic open set of $\mathcal{B}$ is also closed in $\mathcal{T}$.

Solution: Consider a rational $q \in \mathbb{Q}$, and let $x \neq q$. If $x \in \mathbb{Q}$, then $\{x\} \subset \mathbb{R} \setminus \{q\}$ is an open neighborhood. Say, $x \in \mathbb{R} \setminus \mathbb{R}$. Since $x_i \to x \neq q$, there is some $n \geq 1$ such that $|x - x_i| < \epsilon := \frac{|x-q|}{2}$. Then, consider $U_n(x)$. Clearly, $U_n(x) \subset \mathbb{R} \setminus \{q\}$. Thus, $\{q\}$ is closed.

Next, consider an open set $U_n(x)$ for some $x \in \mathbb{R} \setminus \mathbb{Q}$ and $n \geq 0$. Again, for each rational point $q \in \mathbb{Q} \cap (\mathbb{R} \setminus U_n(x))$, we can take $\{q\}$ as an open neighborhood disjoint from $U_n(x)$. Say, $y \neq x$ is an irrational point. Set $\epsilon_0 := \frac{|y-x|}{2}$. Since $x_i \to x$, there is an integer $p \geq 1$ such that

$$|x - x_i| < \epsilon_0.$$

Set

$$\epsilon_1 := \min \frac{1}{2} \{|y - x|, |y - x_1|, \ldots, |y - x_p|\},$$

which is positive as each x_i is rational. Since $y_i \to y$, there is an integer $q \geq 1$ such that for all $i \geq q$ we have

$$|y - y_i| < \epsilon_1.$$

Then, $U_n(x) \cap U_q(y) = \emptyset$. Thus, $U_n(x)$ is closed.

c) The space $X = (\mathbb{R}, \mathcal{T})$ is $T_{3\frac{1}{2}}$, but not T_4.

Hint: Use Jones' lemma.

Solution: It follows that $\mathcal{B}$ is basis of clopen sets in X. Hence, X is completely regular. We show that X is T_0. Let $x \neq y \in \mathbb{R}$. If, without loss of generality, $x \in \mathbb{Q}$, then $\{x\}$

is an open neighborhood, that does not contain y. Suppose, $x, y \in \mathbb{R} \setminus \mathbb{Q}$. Then, for any $n \geq 0$ we have $U_n(x)$ is an open neighborhood, which does not contain y. Thus, X is T_0, and hence, $T_{3\frac{1}{2}}$.

Let us show that X is not normal (and hence, not T_4). Consider $\mathbb{Q}$. For any $x \in \mathbb{R}$, any basic open set contains some rational. Thus, $\mathbb{Q}$ is dense in X. Also, $\mathbb{Q}$ being union of basic open sets, is open, and hence, $\mathcal{I} = \mathbb{R} \setminus \mathbb{Q}$ is closed. For each $x \in \mathbb{I}$, we have $\mathbb{I} \cap U_0(x) = \{x\}$. Thus, $\mathbb{I}$ is a closed, discrete set. Since $\mathbb{Q}$ is countable, and $\mathbb{I}$ is uncountable, it follows by Jones lemma that X is not normal. Thus, X is not T_4.

Q10. Show that the product of a compact space and a paracompact space is again paracompact.

Hint: Use the tube lemma.

Solution: Let X be a paracompact space, and Y be a compact space. Consider an arbitrary open cover $\mathcal{O} = \{O_i\}_{i \in I}$ of $X \times Y$. For each $x \in X$, we have $\{x\} \times Y$ is a compact subspace of $X \times Y$. Hence, there is a finite set $I_x \subset I$ such that

$$\{x\} \times Y \subset \bigcup_{i \in I_x} O_i.$$

By the tube lemma, there is some open neighborhood $x \in U_x \in X$ such that

$$U_x \times Y \subset \bigcup_{i \in I_x} O_i.$$

Now, $\mathcal{U} = \{U_x\}_{x \in X}$ is an open cover of X, which is paracompact. Hence, there is a locally finite refinement, say, $\mathcal{V} = \{V_x\}_{x \in X}$ such that $V_x \subset U_x$ for all $x \in X$. Consider the collection of open sets

$$\mathcal{W} = \{(V_x \times Y) \cap O_i \mid i \in I_x, \ x \in X\}.$$

Let us show that it is a cover of $X \times Y$. Say, $(x, y) \in X$. Then, there is some $x' \in X$ (possibly different from x), such that $x \in V_{x'}$. Then, $(x, y) \in V_{x'} \times Y \subset \bigcup_{i \in I_{x'}} O_i$. Clearly, there is some $i \in I_{x'}$ so that $(x, y) \in (V_{x'} \times Y) \cap O_i$. Thus, $\mathcal{W}$ is a cover, which is a refinement of $\mathcal{O}$ by construction. Next, we show that $\mathcal{W}$ is locally finite. Since $\mathcal{U}$ is a locally finite cover of X, there is some open neighborhood $x \in N \subset X$, and a finite set $F \subset X$ such that

$$N \cap V_x = \emptyset, \ x \in X \setminus F.$$

Suppose $(u, v) \in (N \times Y) \cap ((V_x \times Y) \cap O_i)$ for some $i \in I_x$ and $x \in X$. Then, $u \in N \cap V_x \Rightarrow x \in F$. Thus, it follows that $N \times Y$ can only intersect the collection

$$\{(V_x \times Y) \cap O_i \mid i \in I_x, \ x \in F\},$$

which is clearly finite. Hence, $\mathcal{W}$ is a locally finite open cover, which refines $\mathcal{O}$. Thus, $X \times Y$ is a paracompact space.

Q11. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *smooth function* if f is (continuously) differentiable infinitely many times. Polynomials are smooth, and so are the trigonometric functions $\sin(x)$, $\cos(x)$ etc. The function $\rho(x) = \begin{cases} e^{-\frac{1}{x}}, \ x \geq 0 \\ 0, \ x \leq 0 \end{cases}$ is also smooth; note that ρ is a (constant) polynomial on $(-\infty, 0)$ but not on all of $\mathbb{R}$.

Denote the n^{th}-derivative of a smooth function f as $f^{(n)} : \mathbb{R} \to \mathbb{R}$; for convenience, set $f^{(0)} = f$. Recall that if for some $n \geq 1$ we have $f^{(n)}$ is identically 0 on an interval (a, b) (possibly unbounded), then f is a polynomial of degree $\leq n - 1$ on (a, b). And conversely, if f is a (nonzero) polynomial of degree d on (a, b), then $f^{(d)}|_{(a,b)}$ is a nonzero constant.

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Suppose, for each $x \in \mathbb{R}$, there is some $n = n_x \geq 0$ such that $f^{(n)}(x) = 0$. The goal is to prove that f must be a polynomial. If you wish, you can try to give some direct proof! Otherwise, for the sake of contradiction, let us assume that f is not a polynomial.

a) Denote

$$\Omega = \bigcup \{U \subset \mathbb{R} \mid U \text{ is open, and } f|_U \text{ is a polynomial}\} .$$

By our assumption, $\Omega \neq \mathbb{R}$.

i) If $\Omega \neq \emptyset$, then justify that one can write $\Omega = \bigcup I_j$, for countably many open intervals (possibly unbounded), which are pairwise disjoint.

Solution: As $\mathbb{R}$ is a separable and locally connected space, any open set can be written as countable union of disjoint open connected components. Since the open connected sets are necessarily intervals, the claim follows.

ii) For any bounded interval $[u, v] \subset \Omega$ with $u < v$, show that $f|_{(u,v)}$ is a polynomial.

Solution: Suppose $[u, v] \subset \Omega$ for some $u < v$. For each $x \in [u, v]$ there is some $(a_x, b_x) \subset \mathbb{R}$ such that $f|_{(a_x,b_x)}$ is a polynomial. Since $[a, b]$ is compact, there are finitely many such intervals, say, $\{(a_{x_i}, b_{x_i})\}_{i=1}^k$ that covers $[a, b]$. Suppose $f|_{(a_{x_i},b_{x_i})}$ is a polynomial of degree d_i. Set $d = \max_{1 \leq i \leq k} \{d_i\}$. Then,

$$f^{(d_i)}|_{(a_{x_i},b_{x_i}=0)} \Rightarrow f^{(d)}|_{(a_{x_i},b_{x_i})} = 0, \quad 1 \leq i \leq k.$$

Thus, $f^{(d)} = 0$ on the union $\bigcup_{i=1}^k (a_{x_i}, b_{x_i}) \supset (a, b)$. Hence, $f|_{(a,b)}$ is a polynomial of degree $\leq d - 1$.

iii) Show that $f|_{I_j}$ is a polynomial for any open interval I_j appearing in the expression of Ω.

Hint: Note that any open interval (bounded or unbounded) can be written as an increasing union of countably many bounded closed intervals.

Solution: For any a, b with $a < b$ we have $(a, b) = \bigcup_{n \geq n_0} [a + \frac{1}{n}, b - \frac{1}{n}]$ for some n_0 large, and also $(a, \infty) = \bigcup [a + \frac{1}{n}, a + n], (-\infty, b) = \bigcup [b - n, b - \frac{1}{n}]$. Thus, any open interval can be written as a countable collection of increasing closed intervals. Without loss of generality, let us write $I = \bigcup_i [a_i, b_i]$, where $a_i < b_i$ and $[a_i, b_i] \subset (a_{i+1}, b_{i+1})$ for all i.

Now, suppose $f|_{(a_1,b_1)}$ is a polynomial of degree, say, d. In particular, $f^{(d)}|_{(a_1,b_1)}$ is a nonzero constant. We show that $f^{(d+1)}|_I = 0$ identically. If not, then for some $x \in I$ we have $f^{(d+1)}(x) \neq 0$. By continuity of $f^{(d+1)}$, we have some $x \in (a, b) \subset I$ such that $f^{(d+1)}|_{(a,b)}$ is nonvanishing. Now, from the above increasing union, we can assume that $[a, b] \subset (a_N, b_N)$ for some $N \geq 1$. By previous part, we have $f|_{(a_N,b_N)}$ is a polynomial of degree, say, m. As $f^{(m+1)}|_{(a_N,b_N)} = 0$, we must have $m + 1 \not< d + 1 \Rightarrow d + 1 \leq m + 1 \Rightarrow d \leq m$. Also, $f^{(m+1)}|_{(a_1,b_1)} = 0$ as $(a_1, b_1) \subset (a_N, b_N)$. Thus, $f|_{(a_1,b_1)}$ is a polynomial of degree $\leq m$, which forces, $m \leq d$. Hence, we have $f|_{(a_N,b_N)}$ is a polynomial of degree d. This contradicts $f^{(d+1)}(x) \neq 0$. We conclude that $f^{(d+1)}|_I$ is zero, and hence, f is a polynomial of degree $\leq d$. In fact, f is a polynomial of degree exactly d, as $f^{(d)}|_{(a_1,b_1)}$ is nonzero.

b) Consider the closed sets $S_n := \{x \mid f^{(n)}(x) = 0\} = \left(f^{(n)}\right)^{-1}(0)$.

i) For any $[a, b]$ with $a < b$, prove that $[a, b] \cap S_{n_0}$ has nonempty interior (in the subspace topology of $[a, b]$) for some n_0.

Solution: We are given that for every $x \in \mathbb{R}$, there is some n such that $f^{(n)}(x) = 0 \Rightarrow x \in S_n$. Thus, $\mathbb{R} = \bigcup S_n$. Then, $[a, b] = \bigcup ([a, b] \cap S_n)$. Now, $[a, b]$ is a compact T_2 space, and hence, a Baire space. As $[a, b] \cap S_n$ is closed, all of them cannot be nowhere dense. Consequently, for some n_0, we must have $[a, b] \cap S_{n_0}$ has nonempty interior (in the subspace topology of $[a, b]$).

ii) Conclude that $\overline{\Omega} = \mathbb{R}$, i.e, Ω is dense in $\mathbb{R}$.

Solution: Fix some $[a, b]$ with $a < b$. Then, for some n_0, we have $[a, b] \cap S_{n_0}$ has nonempty intersection. In particular, we can have some $c < d$ such that $(c, d) \subset [a, b] \cap S_{n_0}$. But then $f^{(n_0)}|_{(c,d)} = 0$ which implies, $f|_{(c,d)}$ is a polynomial of degree $\leq n_0 - 1$. Thus, $(c, d) \subset \Omega$. Hence, $(a, b) \cap \Omega \neq \emptyset$. Thus, $\overline{\Omega} = \mathbb{R}$.

c) Denote $X = \mathbb{R} \setminus \Omega$. Note that $X \neq \emptyset$, and the (finite) endpoints of each I_j appearing in Ω belongs to X.

i) Show that any $x \in X$ is *not* an isolated point of X, and hence, there are $x_i \in X$ with $x_i \neq x$, such that $x_i \to x$.

Solution: If possible, suppose $x \in X$ is an isolated point. Then, there are $a < x < b$ such that $(a, b) \cap X = \{x\}$. Consequently, $(a, x) \cup (x, b) \subset \Omega$. Then, there are two open intervals, say, I_j and I_k, such that $(a, x) \subset I_j$ and $(x, b) \subset I_k$. Now, $f|_{I_j}$ and $f|_{I_k}$ are both polynomials, of degree, say, n_1 and n_2. Fix some $n > \max\{n_1, n_2\}$. Then,

$$f^{(n)}|_{(a,x)} = 0 = f^{(n)}|_{(x,b)}.$$

Continuity of $f^{(n)}$ forces that $f^{(n)}(x) = 0$. But then $f^{(n)}|_{(a,b)} = 0$, which implies, $f|_{(a,b)}$ is a polynomial of degree $\leq n - 1$. Then, $x \in (a, b) \subset \Omega$, a contradiction. Thus, for any $x \in X$ we have $x_i \in X$, with $x_i \neq x$, such that $x_i \to x$.

ii) Show that $X \cap S_{n_0}$ has nonempty interior (in the subspace topology of X) for some n_0. Suppose, $X \cap (a_0, b_0) \subset X \cap S_{n_0}$ for some $a_0 < b_0$.

Solution: As $X = \mathbb{R} \setminus \Omega$ is closed in the complete space $\mathbb{R}$, we have X is complete, and hence, a Baire space. As S_n is a cover, again we have some n_0 so that $X \cap S_{n_0}$ has nonempty interior (in the subspace topology of X). That is, we have some $a_0 < b_0$ so that $X \cap (a_0, b_0) \subset X \cap S_{n_0}$.

iii) Show that $f^{(m)}(x) = 0$ for all $m \geq n_0$ and for all $x \in (a_0, b_0) \cap X$.

Hint: By assumption, the limit $f^{(n+1)}(x) = \lim_{h \to 0} \frac{f^{(n)}(x+h) - f^{(n)}(x)}{h}$ exists. For $x_i \to x$ with $x_i \neq x$, one can then consider $h_i := x_i - x \to 0$ in the limit.

Solution: Let $x \in X \cap (a_0, b_0)$. Then, there are $x_i \in X \cap (a_0, b_0)$ with $x_i \neq x$ such that $x_i \to x$. Now, $f^{(n_0)}(x_i) = 0 = f^{(n_0)}(x)$, as $X \cap (a_0, b_0) \subset X \cap S_{n_0}$. Since $f^{(n_0)}$ is differentiable, we have

$$f^{(n_0+1)}(x) = \lim_i \frac{f^{(n_0)}(x_i) - f^{(n_0)}(x)}{x_i - x} = 0.$$

Thus, for all $x \in X \cap (a_0, b_0)$ we have $f^{(n_0+1)}(x) = 0$. In other words, $X \cap (a_0, b_0) \subset X \cap S_{n_0+1}$. Inductively, it follows that for any $m \geq n_0$ we have $X \cap (a_0, b_0) \subset X \cap S_m$, i.e, $f^{(m)}(x) = 0$ for all $x \in X \cap (a_0, b_0)$ and $m \geq n_0$.

iv) Show that for any I_j appearing in Ω, with $I_j \cap (a_0, b_0) \neq \emptyset$, we have $f|_{I_j}$ is a polynomial of degree $\lneq n_0$.
Hint: (a_0, b_0) must contain some end-point of I_j.

Solution: As $\overline{\Omega} = \mathbb{R}$, we must have $(a_0, b_0) \cap \Omega \neq \emptyset$. Now, suppose for some I_j we have $I_j \cap (a_0, b_0) \neq \emptyset$. Clearly, $(a_0, b_0) \subset I_j$ is not possible, as (a_0, b_0) also intersects $X = \mathbb{R} \setminus \Omega$. Then, we must have that some endpoint (left or right) of I_j belongs to (a_0, b_j). Suppose, the endpoint is some x. Then, $x \in X$ as the interval

I_j is maximal (being connected components of Ω). Suppose $f|_{I_j}$ is a polynomial of degree d. Then, $f^{(d)}$ is a nonzero constant, say, c on I_j. By continuity, we must have $f^{(d)}(x) = c$. But we have seen $f^{(m)}(x) = 0$ for all $m \geq n_0$. Hence, we must have $d < n_0$. Thus, whenever (a_0, b_0) intersects some $I_j \subset \Omega$, we have $f|_{I_j}$ is a polynomial of degree $\leq n_0 - 1$.

v) Conclude that f is a polynomial.

Solution: For $x \in X \cap (a_0, b_0)$ we have proved $f^{(n_0)}(x) = 0$. Also, for any $x \in \Omega \cap (a_0, b_0)$, we have $x \in I_j \cap (a_0, b_0)$ for some j, and hence, $f^{(n_0)}(x) = 0$. Thus, $f^{(n_0)}|_{(a_0,b_0)} = 0$. This means f is a polynomial of degree $\leq n_0 - 1$, and so, $(a_0, b_0) \subset \Omega$. This contradicts $x \in X \cap (a_0, b_0)$. Hence, we must have f is a polynomial.