Makeup Examination

Course : Topology (KSM1C03)

Instructor : Aritra Bhowmick

17th January, 2026

Time: 09:30 - 12:30 **Total marks:** 65

Attempt **any** question. Maximum marks will be capped in accordance with the make-up examination rules and your continuous assessment marks.

Q1. Suppose X is a topological space. Show that the topology is indiscrete if and only if given any space Y, any function $f : Y \to X$ is continuous.

Solution: Let X be an indiscrete space. So, the only open sets of X is $\emptyset$ and X. For any function $f : Y \to X$, we clearly have, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(X) = Y$, both of which are open in Y. Thus, f is continuous.

Next, suppose any function $f : Y \to X$ is continuous. Let us consider Y to be X with the indiscrete topology, and the identity function $\mathrm{Id} : Y \to X$. Suppose $U \subset X$ is open in X. Since the identity function is continuous, it follows that $U = \emptyset$ or $U = X$. Thus, X has indiscrete topology.

Q2. A space X is called *scattered* if every nonempty subspace $Y \subset X$ contains an isolated point.

a) Show that a scattered, T_1-space is totally disconnected.

Solution: Let $C \subset X$ be a connected component. By hypothesis, C has an isolated point, say, $x \in C$. Then, $\{x\}$ is open in C. Since X is T_1, we have $\{x\}$ is closed as well. As C is connected, we must have $C = \{x\}$. Thus, every connected component of X is singleton. Hence, X is totally disconnected.

b) Is it possible to remove the T_1 assumption?

Solution: Consider $X = \{-1, 0, 1\}$ with the topology $\mathcal{T} = \{\emptyset, X, \{0\}, \{0, 1\}, \{0, -1\}\}$. Clearly, X is not T_1, as $\{0\}$ is not closed. We check that X is scattered.

i) In X, the point $\{0\}$ is isolated.

ii) For $\{0, 1\}$ and $\{0, -1\}$, again 0 is isolated. The subset $\{1, -1\}$ is a discrete subspace, and hence, both points are isolated.

iii) For any singleton subset, the only element is necessarily isolated.

Thus, X is scattered. But X is not totally disconnected, since X itself is connected (as it has no non-trivial clopen set).

Q3. Given $A, B \subset X$, prove the following.

a) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.

Solution: We have, the open set $\operatorname{int}(A) \cap \operatorname{int}(B) \subset A \cap B$. Since interior of a set is the largest open set contained in it, it follows that $\operatorname{int}(A) \cap \operatorname{int}(B) \subset \operatorname{int}(A \cap B)$. Conversely, $A \cap B \subset A \Rightarrow \operatorname{int}(A \cap B) \subset \operatorname{int}(A)$, and similarly, $\operatorname{int}(A \cap B) \subset \operatorname{int}(B)$. Hence, we have $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.

b) $\operatorname{int}(A \cup B) \supset \operatorname{int}(A) \cup \operatorname{int}(B)$.

Solution: Since $A \subset A \cup B$, we have $\operatorname{int}(A) \subset \operatorname{int}(A \cup B)$. Similarly, we have $\operatorname{int}(B) \subset \operatorname{int}(A \cup B)$. Thus, $\operatorname{int}(A \cup B) \supset \operatorname{int}(A) \cup \operatorname{int}(B)$.

Given an example when $\operatorname{int}(A \cup B) \neq \operatorname{int}(A) \cup \operatorname{int}(B)$.

Solution: Consider $X = \mathbb{R}$ with standard topology. Let $A = (0, 1], B = [1, 2)$. Then, $\operatorname{int}(A) = (0, 1)$ and $\operatorname{int}(B) = (1, 2)$. On the other hand, $\operatorname{int}(A \cup B) = \operatorname{int}((0, 2)) = (0, 2)$. Clearly, $\operatorname{int}(A) \cup \operatorname{int}(B) \neq \operatorname{int}(A \cup B)$.

Q4. Suppose $f : X \to Y$ is a continuous surjection. If X is Lindelöf, show that Y is Lindelöf.

Solution: Let $\{U_\alpha\}$ be an open cover of Y. Since $f : X \to Y$ is continuous, we have $f^{-1}(U_\alpha)$ is open in X. Since f is surjective, we have an open cover $\{f^{-1}(U_\alpha)\}$ of X. Since X is given to be Lindelöf, we have a countable sub-cover, say, $\{f^{-1}(U_{\alpha_i})\}$. We claim $\{U_{\alpha_i}\}$ is a cover of Y. Indeed, for any $y \in Y$, there is some $x \in X$ such that $f(x) = y$. Now, $x \in f^{-1}(U_{\alpha_{i_0}})$ for some α_{i_0}, which means $y = f(x) \in U_{\alpha_{i_0}}$. Thus, the cover $\{U_\alpha\}$ has a countable sub-cover. Hence, Y is Lindelöf.

Q5. Consider the space $X = \mathbb{R} \times \{0, 1\}$ with the dictionary order, and the induced order topology.

a) Show that the subspace $Y = \mathbb{R} \times \{1\} \subset X$ is homeomorphic to the Sorgenfrey line $\mathbb{R}_\ell$ (i.e, $\mathbb{R}$ with the lower limit topology).

Solution: Let us consider the map $f : \mathbb{R}_\ell \to Y$ given by $f(x) = (x, 1)$. Clearly, f is a bijection. Let us show that f is continuous. For any $a, bin\mathbb{R}$ with $a < b$, we have

$$((a,0),(b,1)) \cap Y = \{(x,1) \mid a \leq x < b\} = f([a,b)).$$

Since f maps basic open sets to open sets of Y, it follows that f is an open map. On the other hand, a basic open set in X is of the following types.

- $((a,0),(b,0))$.
- $((a,0),(b,1))$.
- $((a,1),(b,0))$.
- $((a,1),(b,1))$.

It is clear that,

- $Y \cap ((a,0),(b,0)) = \{(x,1) \mid a \leq x < b\}$.
- $Y \cap ((a,0),(b,1)) = \{(x,1) \mid a \leq x < b\}$.
- $Y \cap ((a,1),(b,0)) = \{(x,1) \mid a < x < b\}$.
- $Y \cap ((a,1),(b,1)) = \{(x,1) \mid a < x < b\}$.

These are basic open sets for the subspace topology on Y. Taking inverse under f, we get either $[a,b)$ or (a,b), both of which are open in $\mathbb{R}_\ell$. Thus, f is continuous as well. Hence, f is a homeomorphism onto Y.

b) Conclude that $\mathbb{R}_\ell$ is T_5 (i.e, a completely normal T_1 space).

Solution: Recall that a linearly ordered space is T_5, and any subspace of a T_5-space is again T_5. Thus, Y is a T_5 space. But as $\mathbb{R}_\ell$ is homeomorphic to Y, it follows that $\mathbb{R}_\ell$ is a T_5-space.

Q6. Let X be an uncountable space, equipped with the co-countable topology. For $A \subset X$, prove the following.

a) A is compact if and only if A is finite.

Solution: Let A be compact. If possible, let $\{a_i\}_{i=1}^\infty$ be infinitely many distinct points in A. Consider the sets

$$U_i = X \setminus \{a_i, a_{i+1}, \dots\},$$

which are open in the co-countable topology. Let $a \in A$ be arbitrary. If $a \neq a_i$ for any i, then clearly $a \in U_i$ for all i. Suppose $a = a_i$. Then, we have $a \in U_{i+1}$. Thus, $\{U_i\}$ is an open cover of A. As A is compact, there is a finite sub-cover, say, $A \subset \bigcup_{j=1}^n U_{i_j}$. Let $K = \max_{1 \leq j \leq n} i_j$. Then, for any $i \geq K$, we have $a_i \notin U_{i_j}$ for all j, and thus, A is not covered by $\left\{U_{i_j}\right\}_{j=1}^n$. Hence, A cannot be infinite.

Conversely, if A is finite, it is clearly compact.

b) A is connected if and only if A is singleton or uncountable.

Solution: Suppose A is connected. If possible, let A is countable, but there are at least two distinct points $x, y \in A$. Let $U_x = \{X \setminus A\} \cup \{x\} = X \setminus (A \setminus \{x\})$ and $U_y = X \setminus \{x\}$. Clearly, both U_x, U_y are open sets, and

$$A \cap U_x = \{x\}\,, \quad A \cap U_y = A \setminus \{x\}\,.$$

Thus, A is disconnected. Hence, A is necessarily singleton, or uncountable.

Conversely, a singleton is clearly connected. Let us assume that A is uncountable. If possible, suppose A is not connected. Then, there are open sets $U, V \subset X$ such that $A = (A \cap U) \sqcup (A \cap V)$, and $\emptyset \neq A \cap U, A \cap V \neq A$. It follows that $X \setminus U, X \setminus V$ are countable sets. Now,

$$A \cap U = A \setminus (A \cap V) = A \setminus V \subset X \setminus U, \quad A \cap V = A \setminus (A \cap U) = A \setminus U \subset X \setminus U.$$

Thus, $A = (A \cap U) \cup (A \cap V) \subset (X \setminus U) \cup (X \setminus V)$, a countable set, which is a contradiction. Hence, A must be connected.

Q7. Show that any compact subset of the one-point compactification $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ of $\mathbb{Q}$ is closed in $\hat{\mathbb{Q}}$.

Hint: If x_n is a sequence of rational points converging to some rational number x, then note that $\{x\} \cup \{x_n \mid n \geq 1\}$ is a compact subset of $\mathbb{Q}$.

Solution: Let $A \subset \widehat{\mathbb{Q}}$ be a compact set. We consider two possibilities. If $\infty \notin A$, then we have A is a subspace of $\mathbb{Q}$. As $\mathbb{Q}$ is a T_2 space, it follows that A is closed in $\mathbb{Q}$. But then A is closed in $\widehat{\mathbb{Q}}$ as well.

Now, suppose that $\infty \in A$. If possible, let $x \in \bar{A} \setminus A$. Then, $x \neq \infty$, as $\infty \in A$. Now, consider the open sets $U_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \mathbb{Q}$. Since $x \in \bar{A}$, there are points $a_n \in U_n \cap A$. Thus, we have a sequence of rational points a_n, converging to a rational point x. Since x cannot be any of a_n, there must be infinitely many a_n, and without loss of generality, we can assume that $\{a_n\}$ are distinct. Next, consider the sets

$$V_k = \{\infty\} \cup \mathbb{Q} \setminus (\{x\} \cup \{a_n \mid n \geq k\})\,.$$

Since each $\{x\} \cup \{a_n \mid n \geq k\}$ is closed compact in $\mathbb{Q}$, it follows that V_k is open in $\widehat{\mathbb{Q}}$. Moreover, V_k forms a cover of A. Now, compactness of A implies that there is a finite sub-cover. Without loss of generality, $A \subset \bigcup_{k=1}^{N} V_k$ for some N large. But then $a_{N+1} \notin V_k$ for $1 \leq k \leq N$, a contradiction. Hence, we must have A is closed.

Q8. Consider the space $X = [0,1] \cup 1^*$, where 1^* is a separate point. Consider the collection

$$\mathcal{B} := \{U \subset [0,1] \mid U \text{ is open in the usual topology}\} \cup \{(a,1) \cup \{1^*\} \mid 0 \le a < 1\}.$$

a) Show that $\mathcal{B}$ is a basis for a topology on X, called the *telophase topology*.

<u>**Solution:**</u> For any $x \in X$, we have some set in $\mathcal{B}$ that contains x. Let $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$. We consider few cases.

- If $x = 1$, then none of B_1, B_2 can be of the form $(a,1) \cup \{1^*\}$. But then we have $B_1 \cap B_2 \in \mathcal{B}$.
- If $x = 1^*$, then we must have $B_1 = (a_1, 1) \cup 1^*$ and $B_2 = (a_2, 1) \cup 1^*$ for some $0 \le a_1, a_2 < 1$. Take $a = \max\{a_1, a_2\}$. Then, $B_1 \cap B_2 = (a,1) \cup \{1^*\} \in \mathcal{B}$.
- Finally, suppose $x \ne 1, 1^*$. But then we can find some interval $x \in J \subset [0,1)$ such that $J \subset B_1 \cap B_2$, and clearly, $J \in \mathcal{B}$.

Thus, $\mathcal{B}$ is a basis for some topology.

b) On $[-1,1]$ define an equivalence relation, : $x \sim y$ if and only if

$$x = y, \qquad \text{or} \qquad -1 < x, y < 1, \text{ and } x = -y.$$

Show that the quotient space $[-1,1]/_\sim$ is homeomorphic to the telophase topology.

<u>**Solution:**</u> Let us define the map $f : [-1,1] \to X$ as follows

$$f(x) = \begin{cases} |x|, & -1 < x < 1, \\ 1, & x = 1, \\ 1^* & x = -1. \end{cases}$$

Clearly, we have an induced map $\tilde{f} : [-1,1]/_\sim \to X$, which is evidently bijective. By property of the quotient topology, $\tilde{f}$ is continuous as well. Finally, in order to prove $\tilde{f}$ is open, we observe that the open sets of $[-1,1]/_\sim$ are images of the following.

- $U \cup -U$ for some $U \subset [0,1)$ open.
- $\{1\} \cup U \cup -U$ for some $U \subset [0,1)$ open.
- $\{-1\} \cup U \cup -U$ for some $U \subset [0,1)$ open.

Then, it can be easily verified that $\tilde{f}$ is open. Thus, $\tilde{f}$ is a homeomorphism.

c) Give an example of two compact sets in the telophase topology, whose intersection is not compact.

<u>**Solution:**</u> Consider $A = [0,1]$ and $B = [0,1) \cup \{1^*\}$ as subspaces of X. Then, $A = \tilde{f}([0,1])$ and $B = \tilde{f}([-1,0])$ is clearly compact. On the other hand, we have

$A \cap B = [0, 1)$, which is not compact in $[0, 1]$ with the usual topology. Thus, $A \cap B$ is not compact in X either.

Q9. Recall, a space X is called *exhaustible by compacts* if there are compact sets $\{K_n\}_{n\geq 1}$ such that $X = \bigcup K_n$ and $K_n \subset \operatorname{int}(K_{n+1})$ for all $n \geq 1$. Suppose X is a T_2 space. Show the following are equivalent.

a) X is exhaustible by compacts.

b) X is Lindelöf and locally compact.

Hint: Recall that finite union of compacts is compact, and interior of finite union contains the finite union of interiors.

Solution: Suppose, $X = \bigcup_{n\geq 1} K_n$ where $K_n \subset X$ are compacts, satisfying $K_n \subset \mathring{K}_{n+1}$. Let $\mathcal{U} = \{U_\alpha\}_\Lambda$ be an arbitrary open cover of X. Since each K_n is compact, for each n, there is a finite subset $\Lambda_n \subset \Lambda$ such that $K_n \subset \bigcup_{\alpha\in\Lambda_n} U_\alpha$. Consider $\Lambda' = \bigcup_{n\geq 1} \Lambda_n$, which is clearly countable. Since $X = \bigcup K_n$, we have $X = \bigcup_{\alpha\in\Lambda'} U_\alpha$. Thus, $\mathcal{U}$ admits a countable sub-cover. In other words, X is Lindelöf. Also, for any $x \in X$, suppose $x \in K_n$, which means $x \in \mathring{K}_{n+1}$. As X is T_2, it follows that X is locally compact.

Conversely, suppose X is locally compact and Lindelöf. For all $x \in X$, there is a compact set C_x such that $x \in \mathring{C}_x$. Now, $X = \bigcup_{x\in X} \mathring{C}_x$ is an open cover. As X is Lindelöf, there is a countable subcover, say, $X = \bigcup_{n\geq 1} \mathring{C}_n$. Now, we inductively construct the exhaustion. Let $K_1 = C_1$. Since the set $K_1 \cup C_2$ is compact, there is an integer $N \geq 1$ such that

$$K_1 \cup C_2 \subset \mathring{C}_1 \cup \cdots \cup \mathring{C}_N.$$

Let $K_2 = \bigcup_{i=1}^N C_i$. Clearly, $\mathring{K}_2 \supset \bigcup_{i=1}^N \mathring{C}_i \supset K_1 \cup C_2$. Inductively, assume that we have defined K_k with $C_k \subset K_k$, and moreover, $K_i \subset \mathring{K}_{i+1}$ holds for $1 \leq i < k$. Now, $K_k \cup C_{k+1}$ is compact, and hence, we have some integer $M \geq 1$ such that

$$K_k \cup C_{k+1} \subset \bigcup_{i=1}^{M} \mathring{C}_i.$$

Consider, $K_{k+1} := \bigcup_{i=1}^M C_i$. Clearly,

$$\mathring{K}_{k+1} \supset \bigcup_{i=1}^{M} \mathring{C}_i \supset K_k \cup C_{k+1}.$$

Since $X = \bigcup C_n \subset \bigcup K_n$, we have $X = \bigcup K_n$. Moreover, $K_n \subset \mathring{K}_{n+1}$ holds by construction. Thus, X is exhaustible by compacts.