End-semester Examination

Course : Topology (KSM1C03)

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Time: 2:00 PM onwards **Total marks:** 100

Attempt any **3** from **Q1 - Q5**, any **3** from **Q6 - Q10**, and **Q11** is mandatory. You can get maximum **70 marks**.

Q1. A topology $\mathcal{T}$ on X is said to be *minimally Hausdorff* if $(X, \mathcal{T})$ is a T_2-space, and given any strictly coarser topology $\mathcal{T}' \subsetneq \mathcal{T}$ on X, we have $(X, \mathcal{T}')$ is not T_2. Show that a compact, T_2 space is minimally Hausdorff. $[5]$

Q2. A space X is called *hereditarily connected* if every subspace of X is connected. Show that X is hereditarily connected if and only if the topology on X is a totally ordered set with respect to set inclusion (i.e., if and only if for any two open sets $U, V \subset X$ we have $U \subset V$ or $V \subset U$). $[2\frac{1}{2} + 2\frac{1}{2} = 5]$

Q3. Let X be a locally connected, separable space. Show that any open set $U \subset X$ can be written as a countable union of disjoint, open, connected sets. $[5]$

Q4. Let X be a locally compact, T_2 space. $[3 + 2 = 5]$

a) Show that X is $T_{3\frac{1}{2}}$.

b) If X is second countable, show that X is paracompact.

Q5. Show that a perfectly normal, T_0-space is T_6. $[5]$

Q6. Let X be a T_2 space. $[3 + (1 + 4) + 2 = 10]$

a) Suppose $f, g : Z \to X$ are continuous maps. Show that the set $E(f, g) := \{z \in Z \mid f(z) = g(z)\}$ is closed in Z.

b) Let $\iota : A \to X$, and $r : X \to A$ be continuous maps satisfying $r \circ \iota = \mathrm{Id}_A$. Show that ι is injective, and $\iota(A)$ is closed in X.

A subspace $A \subset X$ is called a *retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for any $a \in A$. Show that a retract of a T_2-space is a closed subset.

Q7. On $\mathbb{R}$, consider the particular point topology $\mathcal{T}_0$ with base 0, i.e,

$$\mathcal{T}_0 := \{\emptyset\} \cup \{A \subset \mathbb{R} \mid 0 \in A\}.$$

Denote $X = (\mathbb{R}, \mathcal{T}_0)$. $[(2 \times 4) + 2 = 10]$

a) Which of the following properties does X have? Justify.

i) Lindelöf ii) Separable iii) Locally compact iv) Path connected.

b) Explicitly describe all the open sets in the Alexandroff compactification $\hat{X} = X \cup \{\infty\}$.

Q8. On $\mathbb{R}$, consider the following topology

$$\mathcal{T} := \{\emptyset, \mathbb{R}\} \cup \{S \mid S \subset \mathbb{R},\ 0 \notin S\} \cup \{\mathbb{R} \setminus C \mid C \subset \mathbb{R} \setminus \{0\} \text{ is countable}\}.$$

The space $X = (\mathbb{R}, \mathcal{T})$ is called the *fortissimo space* on $\mathbb{R}$. $[3 + 3 + 2 + 2 = 10]$

a) Show that X is T_5.

b) Show that X is not T_6.

c) Show that X is Lindelöf, but not compact.

d) Is X metrizable?

Q9. On $\mathbb{R}$, for each irrational x, fix a sequence $x_i \in \mathbb{Q}$ such that $x_i \to x$ (in the usual sense). Denote the set

$$U_n(x) = \{x\} \cup \{x_i \mid i > n\}, \quad x \in \mathbb{R} \setminus \mathbb{Q},\ n \geq 0.$$

Consider the collection of subsets

$$\mathcal{B} := \{\{q\} \mid q \in \mathbb{Q}\} \cup \{U_n(x) \mid x \in \mathbb{R} \setminus \mathbb{Q},\ n \geq 0\}.$$

Prove the following. $[2 + 3 + (2 + 3) = 10]$

a) $\mathcal{B}$ is a basis for a topology, say, $\mathcal{T}$ on $\mathbb{R}$ (called the *rational sequence topology*).

b) Each basic open set of $\mathcal{B}$ is also closed in $\mathcal{T}$.

c) The space $X = (\mathbb{R}, \mathcal{T})$ is $T_{3\frac{1}{2}}$, but not T_4.

Hint: Use Jones' lemma.

Q10. Show that the product of a compact space and a paracompact space is again paracompact. $[10]$

Hint: Use the tube lemma.

Q11. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *smooth function* if f is (continuously) differentiable infinitely many times. Polynomials are smooth, and so are the trigonometric functions $\sin(x)$, $\cos(x)$ etc. The function $\rho(x) = \begin{cases} e^{-\frac{1}{x}},\ x \geq 0 \\ 0,\ x \leq 0 \end{cases}$ is also smooth; note that ρ is a (constant) polynomial on $(-\infty, 0)$ but not on all of $\mathbb{R}$.

Denote the n^{th}-derivative of a smooth function f as $f^{(n)} : \mathbb{R} \to \mathbb{R}$; for convenience, set $f^{(0)} = f$. Recall that if for some $n \geq 1$ we have $f^{(n)}$ is identically 0 on an interval (a, b) (possibly unbounded), then f is a polynomial of degree $\leq n - 1$ on (a, b). And conversely, if f is a (nonzero) polynomial of degree d on (a, b), then $f^{(d)}|_{(a,b)}$ is a nonzero constant.

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Suppose, for each $x \in \mathbb{R}$, there is some $n = n_x \geq 0$ such that $f^{(n)}(x) = 0$. The goal is to prove that f must be a polynomial. If you wish, you can try to give some direct proof! Otherwise, for the sake of contradiction, let us assume that f is not a polynomial. $[25]$

a) Denote

$$\Omega = \bigcup \{U \subset \mathbb{R} \mid U \text{ is open, and } f|_U \text{ is a polynomial}\}.$$

By our assumption, $\Omega \neq \mathbb{R}$.

i) If $\Omega \neq \emptyset$, then justify that one can write $\Omega = \bigcup I_j$, for countably many open intervals (possibly unbounded), which are pairwise disjoint.

ii) For any bounded interval $[u, v] \subset \Omega$ with $u < v$, show that $f|_{(u,v)}$ is a polynomial.

iii) Show that $f|_{I_j}$ is a polynomial for any open interval I_j appearing in the expression of Ω.

Hint: Note that any open interval (bounded or unbounded) can be written as an increasing union of countably many bounded closed intervals. $[1 + 2 + 3 = 6]$

b) Consider the closed sets $S_n := \{x \mid f^{(n)}(x) = 0\} = \left(f^{(n)}\right)^{-1}(0)$.

i) For any $[a, b]$ with $a < b$, prove that $[a, b] \cap S_{n_0}$ has nonempty interior (in the subspace topology of $[a, b]$) for some n_0.

ii) Conclude that $\overline{\Omega} = \mathbb{R}$, i.e, Ω is dense in $\mathbb{R}$. $[1 + 3 = 4]$

c) Denote $X = \mathbb{R} \setminus \Omega$. Note that $X \neq \emptyset$, and the (finite) endpoints of each I_j appearing in Ω belongs to X.

i) Show that any $x \in X$ is *not* an isolated point of X, and hence, there are $x_i \in X$ with $x_i \neq x$, such that $x_i \to x$.

ii) Show that $X \cap S_{n_0}$ has nonempty interior (in the subspace topology of X) for some n_0. Suppose, $X \cap (a_0, b_0) \subset X \cap S_{n_0}$ for some $a_0 < b_0$.

iii) Show that $f^{(m)}(x) = 0$ for all $m \geq n_0$ and for all $x \in (a_0, b_0) \cap X$.

Hint: By assumption, the limit $f^{(n+1)}(x) = \lim_{h \to 0} \frac{f^{(n)}(x+h) - f^{(n)}(x)}{h}$ exists. For $x_i \to x$ with $x_i \neq x$, one can then consider $h_i := x_i - x \to 0$ in the limit.

iv) Show that for any I_j appearing in Ω, with $I_j \cap (a_0, b_0) \neq \emptyset$, we have $f|_{I_j}$ is a polynomial of degree $\lneq n_0$.

Hint: (a_0, b_0) must contain some end-point of I_j.

v) Conclude that f is a polynomial. $[3 + 1 + 3 + 4 + 4 = 15]$

Definitions/Hints

- A relation $\leq$ on a set X is called a *total order* if the following holds for any $x, y, z \in X$.
  a) For any $x \in X$ we have $x \leq x$.
  b) For any $x, y, z \in X$ we have $x \leq y, \quad y \leq z \Rightarrow x \leq z$.
  c) For any $x, y \in X$ we have $x \leq y, \quad y \leq x \Rightarrow x = y$.
  d) For any $x, y \in X$ we have either $x \leq y$ or $y \leq x$.
- X is called *locally connected* if given any open set $U \subset X$ and a point $x \in U$, there is an open neighborhood $x \in V \subset U$, such that V is a connected set.
- X is *separable* if there is a countable dense subset.
- X is *second countable* if there is a countable basis.
- X is *locally compact* if given any open set $U \subset X$ and a point $x \in U$, there is a compact set $C \subset X$ such that $x \in \mathring{C} \subset C \subset U$, where $\mathring{C}$ is the interior of C.
- If X is T_2, then X is locally compact if and only if for any open set U and any $x \in U$, we have $x \in V \subset \bar{V} \subset U$, with V open and $\bar{V}$ compact.
- X is T_0 if given any two distinct points $x, y \in X$, there is an open set $U \subset X$ which contains exactly one of $\{x, y\}$.
- X is T_1 if given any two distinct points $x, y \in X$ there are open sets $U, V \subset X$ such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Equivalently, any singleton subsets of X is closed.
- X is *completely regular* if given any closed set $A \subset X$ and a point $x \in X \setminus A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.
- X is $T_{3\frac{1}{2}}$ if X is completely regular and T_0.
- X is *normal* if any of the following holds:
  a) Given closed subsets $A, B \subset X$ with $A \cap B = \emptyset$, there are open set $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.
  b) Given any closed set $A \subset X$, and an open set $U \subset X$ with $A \in U$, there is an open set $V \subset X$ such that $A \in V \subset \bar{V} \subset U$.
  c) Given closed subsets $A, B \subset X$ with $A \cap B = \emptyset$, there is a continuous map $f : X \to [0, 1]$ such that $f(A) = 0, f(B) = 1$.
- X is T_4 if it is normal and T_1.
- X is *completely normal* if any of the following holds:
  a) Any subset $A \subset X$ is normal.
  b) Any open subset $U \subset X$ is normal.
  c) Given any subsets $A, B \subset X$, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.
- X is T_5 if it is completely normal and T_1.
- X is *perfectly normal* if any of the following holds:
  a) Given closed subsets $A, B \subset X$ with $A \cap B = \emptyset$, there is a continuous function $f : X \to [0, 1]$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.
  b) X is normal, and any closed set $C \subset X$ is a G_δ-set.
  c) Given any closed set $C \subset X$, there is a continuous function $f : X \to \mathbb{R}$ such that $C = f^{-1}(0)$.
- X is T_6 if it is perfectly normal and T_1.
- X is *Lindelöf* if any open cover has a countable sub-cover.

- X is paracompact if given any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ there exists a locally finite open cover $\mathcal{V} = \{V_i\}_{j \in J}$ such that for each $j \in I$ there is some $i \in I$ such that $V_j \subset U_i$. One can assume that $J = I$ and $V_i \subset U_i$ for all $i \in I$.
- A collection $\mathcal{A} = \{A_\alpha \subset X\}$ of subsets is called *locally finite*, if for any $x \in X$, there exists an open neighborhood $x \in U \subset X$, such that U intersects at most finitely many (possibly none) of A_α.
- X is a *Baire space* if countable intersection of open dense sets of X is again dense. Equivalently, countable union of closed nowhere dense sets of X has empty interior.
- If X is locally compact, T_2, then X is a Baire space.
- If X is completely metrizable, then X is a Baire space.
- Tube lemma : Let $x \in X$ and $C \subset Y$ be compact. If $\{x\} \times C \subset O \subset X \times Y$, where O is open, then there exists an open neighborhood $x \in U \subset X$ such that $\{x\} \times C \subset U \times C \subset O \subset X \times Y$.
- Jones' lemma : Let $Q \subset X$ be a dense set, and $Z \subset X$ be a closed discrete set. If $|\mathcal{P}(Q)| < |\mathcal{P}(Z)|$, then X cannot be normal.