Assignment 9

Topology (KSM1C03)

Submission Deadline: 15th November, 2025

1) Let X_α be a collection of spaces, and $X = \prod X_\alpha$ is the product space. Show that X is completely regular if and only if each X_α is completely regular. Conclude that X is Tychonoff if and only if each X_α is Tychonoff.

   $8 + 2 = 10$

2) Let $f : X \to Y$ be closed, continuous, surjective map. If X is normal, show that Y is normal. Conclude that closed continuous image of a T_4-space is again T_4.

   $8 + 2 = 10$

3) Let $(X, \leq)$ be a totally ordered space, with the induced order topology. Suppose X is connected, and $|X| \geq 2$. Show that X cannot be countable.

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4) *Jones' Lemma:* Suppose X has a dense subspace $Q \subset X$, and closed discrete subspace $Z \subset X$, with $2^{|Q|} < 2^{|Z|}$ (equivalently, $|\mathcal{P}(Q)| < |\mathcal{P}(Z)|$). Prove that X is not normal.

   Hint : Suppose X is normal. Given any subspace $S \subset Z$, try to separate S and $Z \setminus S$ by open sets $S \subset U_S, Z \setminus S \subset V_S$. Argue that $S \mapsto U_S \cap Q$ is an injective map. This implies $|\mathcal{P}(Z)| \leq |\mathcal{P}(Q)|$, a contradiction.

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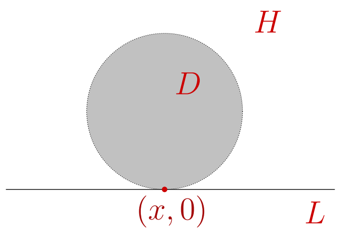
5) Show that the Sorgenfrey plane (i.e, the product space $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$) is $T_{3\frac{1}{2}}$ but not T_4.

   Hint : Try to show that the Sorgenfrey plane (or the line) is zero-dimensional (i.e, admits a basis of clopen sets), is separable, and the diagonal $\{(x, -x) \mid x \in \mathbb{R}\} \subset \mathbb{R}_\ell \times \mathbb{R}_\ell$ is a closed discrete subspace.

   $5 + 5 = 10$

6) Consider $H = \{(x, y) \in \mathbb{R} \mid y > 0\}$ and $L = \{(x, 0) \in \mathbb{R}^2 \mid \ \in \mathbb{R}\}$. For points of H, the usual open neighborhoods from H forms a neighborhood basis. For a point $(x, 0) \in L$, a basic open neighborhood is of the form $\{x\} \cup D$, where $D \subset H$ is an open disc (in the usual sense), such that the closed disc $\bar{D}$ (in $\mathbb{R}^2$) touches the line L tangentially at $(x, 0)$.



This space is called the *Moore plane* (or the *Nemytskii plane*).

a) Show that the Moore plane is $T_{3\frac{1}{2}}$.

Hint : For any point $p \in H$, consider a euclidean disc B centered at p. Define $f : X \to [0, 1]$ by setting $f(p) = 0, f(X \setminus B) = 1$, and radially interpolating in between. For $p \in L$, consider basic open set $U = \{x\} \cup D$ as above. Again, define $f(p) = 0, f(X \setminus U) = 1$, and interpolate via straight lines from the points on the circle to the point p. Check that f is continuous. Conclude that X is completely regular.

b) Show that the Moore plane is not T_4.

Hint : X is separable, and L is a closed discrete subspace.

$6 + 4 = 10$

7) Suppose X a regular, Lindelöf space. Show that X is normal.

Hint : Get suitable countable covers $A \subset \bigcup_{i=1}^{\infty} U_{a_i}$ and $B \subset \bigcup_{i=1}^{\infty} V_{b_i}$ using regularity and Lindelöf. Define $U_n = U_{a_n} \setminus \bigcup_{i \leq n} \overline{V_{a_i}}, V = V_{a_n} \setminus \bigcup_{i \leq n} \overline{U_{a_i}}$. Justify that $U := \bigcup U_n, V := \bigcup V_n$ is a separation by opens.

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8) Suppose X is a regular, hereditarily Lindelöf space. Show that X is perfectly normal.

Hint : Show that given closed set C is G_δ by proving that $U = X \setminus C$ is F_σ (i.e, countable union of closed sets).

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