Assignment 8

Topology (KSM1C03)

Submission Deadline: 4th November, 2025

1) A metric space (X, d) is said to have the *Lebesgue number property* if given any open cover $\mathcal{U} = \{U_\alpha\}$ of X, there exists a number $\delta = \delta(\mathcal{U}) > 0$, which is a Lebesgue number for the covering (i.e, given any subset $A \subset X$, with $\mathrm{Diam} A < \delta$, there is some $U_\alpha \in \mathcal{U}$ such that $A \subset U_\alpha$). Suppose (X, d) has the Lebesgue number property. Show that every continuous map $f : X \to Y$, where Y is a metric space, is uniformly continuous. We shall see later that the converse is also true!

10

2) Given a space X, show that the following are equivalent.

   a) X is completely T_2.

   b) The map

$$\iota : X \longrightarrow [0,1]^{C(X,[0,1])} := \prod_{f \in C(X,[0,1])} [0,1]$$
$$x \longmapsto \big(f(x)\big)_{f \in C(X,[0,1])}$$

   is injective, where $C(X, [0, 1]) = \{f : X \to [0, 1] \mid f \text{ is continuous}\}$.

$5 + 5 = 10$

3) An open set $O \subset X$ is called a *regular open set* if it satisfies $\mathrm{int}(\bar{O}) = O$. A space X is called *semiregular* if it admits a basis $\mathcal{B}$ of regular open sets. Prove the following.

   a) A regular space is always semiregular.

   b) A semiregular space may not be regular. (Hint : Arens square)

   c) A semiregular, T_2 space may not be $T_{2\frac{1}{2}}$ (and hence, not functionally T_2 either). (Hint: the double-origin plane)

$4 + 4 + 2 = 10$

4) Let us verify the usual operations on regular spaces.

   a) Show that a subspace of a regular space is regular (that is, regularity is a hereditary property).

   b) Let $\{X_\alpha\}$ be a collection of (nonempty) spaces, and $X = \prod X_\alpha$ be the product space. Show that X is regular if and only if each X_α is regular.

We shall see later that continuous image of a regular space need not be regular.

$4 + 6 = 10$

5) Given $K = \left\{ \frac{1}{n} \mid n \geq 1 \right\}$, recall the topology $\mathbb{R}_K$ on the reals : every usual open set of $\mathbb{R}$ is open in $\mathbb{R}_K$, and moreover, for any usual open set $U \subset \mathbb{R}$, sets of the form $U \setminus K$ is also open. Show that $\mathbb{R}_K$ is functionally T_2 (hence $T_{2\frac{1}{2}}$), but not T_3.

   Hint : Show that $\mathbb{R}_K$ is submetrizable (since the identity map $\mathbb{R}_K \to \mathbb{R}$ is continuous). Also, note that K is closed in $\mathbb{R}_K$.

$4 + 6 = 10$

6) On the set $[0, 1)$ consider the following topology

$$\mathcal{T} := \{\emptyset\} \cup \{[0, 1) \setminus F \mid F \subset (0, 1) \text{ is finite}\} \cup \{S \mid S \subset (0, 1)\}.$$

   Let $X = ([0, 1), \mathcal{T})$ be the space.

   a) Show that X is the one-point compactification of $\mathbb{R}$ equipped with discrete topology.

   b) Suppose $f : X \to \mathbb{R}$ is a continuous map (where $\mathbb{R}$ has the usual topology). Show that f is constant outside a countable subset of $(0, 1)$.

      Hint : Note that
$$\{f(0)\} = \bigcap_{n \geq 1} \left(f(0) - \frac{1}{n}, f(0) + \frac{1}{n} \right),$$
      and look at $f^{-1}(f(0))$.

$4 + 6 = 10$

7) A space X is called *zero-dimensional* if it admits a basis of clopen sets (i.e, both open and closed sets).

   a) Show that a zero-dimensional space is completely regular.

   b) Show that $[0, \Omega] = \overline{S_\Omega}$ is zero-dimensional. (Hint : if $\alpha = \beta + 1$ for some β, then $(\beta, \beta + 2) = \{\alpha\}$ is clopen. What if there is no such β?).

   c) Show that arbitrary product of zero-dimensional spaces is again zero-dimensional.

   d) Conclude that the Tychonoff plank is a Tychonoff space.

$3 + 4 + 4 + 4 = 15$

8) The *Thomas plank* is defined as the product $[0, 1) \times \left(\{0\} \cup \left\{ \frac{1}{n} \mid n \geq 1 \right\}\right)$, where $[0, 1)$ is the fort space on the reals, and $K = \{0\} \cup \left\{ \frac{1}{n} \mid n \geq 1 \right\}$ has the subspace topology from $\mathbb{R}$ (equivalently, K is the Fort space of $\mathbb{N}$). The *deleted Thomas plank* is defined by deleting the point $\{(0, 0)\}$ from the Thomas plank.
   Construct the *Thomas corkscrew* : take four copies of the deleted Thomas plank to make a coordinate plane (by reflecting them as necessary), add two special points $\{\alpha_\pm\}$, and finally, perform the corkscrew construction.
   Show that the Thomas corkscrew is T_3, but not $T_{3\frac{1}{2}}$.

$10 + (5 + 5) = 20$