Assignment 7

Topology (KSM1C03)

Submission Deadline: 24^{th} October, 2025

1) A set $A \subset X$ is called a *G_δ-set* if we can write A as the intersection of countably many open sets of X.
   a) Suppose X is a first countable, T_1-space. Show that every singleton of X is G_δ.
   b) Show that the irrationals $\mathbb{R} \setminus \mathbb{Q}$ form a G_δ-set in $\mathbb{R}$.

$5 + 5 = 10$

2) Suppose (X, d) is a metric space. For any $A \subset X$ and any $\epsilon > 0$, define the ϵ-neighborhood of A as
$$\mathcal{N}_\epsilon(A) := \{x \in X \mid d(a, x) < \epsilon \text{ for some } a \in A\}.$$
   a) Show that $\mathcal{N}_\epsilon(A)$ is an open set containing A.
   b) Show that any closed subset $C \subset X$ is a G_δ-set.

   Note: A space X is called a *G_δ-space* if every closed subset $C \subset X$ is a G_δ-set. Thus, any metric space is a G_δ-space.

$5 + 5 = 10$

3) Suppose $f : X \to (Y, d)$ is a function (not necessarily continuous). A point $x \in X$ is called a *point of continuity* of f if for any $\epsilon > 0$, there exists an open neighborhood $x \in U \subset X$, such that $f(U) \subset B_d(f(x), \epsilon)$. Show that the set of points of continuity of f is a G_δ-set.

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4) Suppose X is a Lindelöf, G_δ-space. Show that X is hereditarily Lindelöf.

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5) Suppose X is a hereditarily Lindelöf, T_2 space. Show that every singleton of X is a a G_δ-set.

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6) Prove that every second countable space is hereditarily Lindelöf.

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7) Suppose X is a second countable space. Let $\mathcal{K} = \{C_\alpha\}$ be a collection of closed subsets of X, such that for any decreasing sequence $C_{\alpha_1} \supset C_{\alpha_2} \supset \dots$ of elements of $\mathcal{K}$, we have $\cap C_{\alpha_i} \in \mathcal{K}$. Show that $\mathcal{K}$ has minimal element $A \in \mathcal{K}$, i.e, no proper subset of A belongs to $\mathcal{K}$.

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