Assignment 6

Topology (KSM1C03)

Submission Deadline: 20th October, 2025

1) A family of subsets $\mathcal{A} \subset \mathcal{P}(X)$ of X is called *inadequate* if $\mathcal{A}$ does not cover X, and is called *finitely inadequate* if any finite sub-collection of $\mathcal{A}$ is inadequate (i.e, does not cover X).

   a) Given a space X, show that the following are equivalent.

      i) X is compact.

      ii) Every finitely inadeqate family of open sets of X is inadequate.

   b) Given a space $(X, \mathcal{T})$, let $\mathcal{S}$ be a finitely inadequate collection of open sets. Consider the collection
   $$\mathfrak{B} := \{\mathcal{B} \subset \mathcal{T} \mid \mathcal{B} \text{ is finitely inadequate, and } \mathcal{S} \subset \mathcal{B}\},$$
   and equip it with the partial order $\mathcal{B}_1 \le \mathcal{B}_2$ if and only if $\mathcal{B}_1 \subset \mathcal{B}_2$.

      i) Show that there exists a maximal element in the poset $(\mathfrak{B}, \le)$.

      ii) Suppose $\mathcal{B}_0$ is a maximal finitely inadequate collection of open sets. If for any collection $U_1, \ldots, U_n \subset X$ of open sets we have $\bigcap_{i=1}^n U_i \subset U \in \mathcal{B}_0$, then show that $U_{i_0} \in \mathcal{B}_0$ for some $1 \le i_0 \le n$.

   c) (Alexander's sub-base lemma) Given a space X, show that the following are equivalent.

      i) X is compact.

      ii) There exists a sub-basis $\mathcal{S}$ of X, such that each cover of X by elements of $\mathcal{S}$ has a finite sub-cover.

      Hint : For i) $\Rightarrow$ ii), just take $\mathcal{S}$ to be the whole topology. For ii) $\Rightarrow$ i), if possible, let $\mathcal{U}$ be an open cover of X, such that there is no finite sub-cover. Using part b), get a maximal finitely inadequate collection $\mathcal{B}_0 \supset \mathcal{U}$. Consider the collection $\mathcal{D} := \mathcal{B}_0 \cap \mathcal{S}$. Note that $\mathcal{D}$ is still finitely deficient, and hence, does not cover X. Let $x_0 \in X \setminus \bigcup_{U \in \mathcal{D}} U$. Get $V \in \mathcal{B}_0$ such that $x_0 \in V$. As $\mathcal{S}$ is a sub-basis, get $B_1, \ldots, B_k \in \mathcal{S}$ such that $x_0 \in \bigcap_{i=1}^k B_i \subset V \in \mathcal{B}_0$. Using part b), we have $B_{i_0} \in \mathcal{B}_0 \Rightarrow B_{i_0} \in \mathcal{D}$. This is a contradiction.

   d) (Tychonoff's theorem) Suppose $\{X_\alpha\}$ is a family of compact spaces, and $X = \Pi_\alpha X_\alpha$ is the product space. Using Alexander's sub-base lemma, show that X is compact.

Hint : Consider the sub-basis

$$\mathcal{S} := \left\{ \pi_\alpha^{-1}(U) \;\middle|\; U \underset{\text{open}}{\subset} X_\alpha. \right\}.$$

Say, $\mathcal{U} \subset \mathcal{S}$ is a cover of X. For each α, consider the collection

$$\mathcal{U}_\alpha := \left\{ U \underset{\text{open}}{\subset} X_\alpha \;\middle|\; \pi_\alpha^{-1}(U) \in \mathcal{U} \right\}.$$

Show that $\mathcal{U}_{\alpha_0}$ is a cover of X_{α_0} for some α_0. If not, using axiom of choice, there is an $x \in X$ such that

$$x_\alpha = \pi_\alpha(x_\alpha) \in X_\alpha \setminus \bigcup_{U \in U_\alpha} U, \qquad \text{for all } \alpha.$$

But then x is not covered by $\mathcal{U}$, a contradiction. As X_{α_0} is compact, we then have a finite sub-cover $U_1, \ldots, U_n \subset X_{\alpha_0}$, and then, $X = \bigcup_{i=1}^n \pi_{\alpha_0}^{-1}(U_i)$ follows. Thus, $\mathcal{U}$ has a finite sub-cover. Conclude the proof by Alexander's sub-base lemma.

$$5 + (4 + 6) + (2 + 8) + 10 = 35$$

2) Suppose X, Y are compact, Y is T_1, and $f : X \to Y$ is a surjective continuous map. Prove that there exists a compact set $X_0 \subset X$ such that $f : X_0 \to Y$ is surjective, but for any proper closed set $C \subsetneq X_0$ we have $f(C) \neq Y$.

Hint : Consider the collection

$$\mathcal{U} := \left\{ U \underset{\text{open}}{\subset} X \;\middle|\; f(X \setminus U) = Y \right\},$$

and apply Zorn's lemma.

$$10$$

3) Show that any totally ordered set $(X, \leq)$ with the least upper bound property is locally compact.

$$10$$

4) Suppose X is a locally compact, T_2 space.

a) If $U \subset X$ is open, and $C \subset X$ is closed, show that $U \cap C$ is a locally compact set.

b) Suppose $Y \subset X$ is a locally compact set. Then, Y is the intersection of an open set and a closed set of X.

Hint : Show that Y is open in $\overline{Y}$ in the subspace topology.

$$4 + 6 = 10.$$

5) Suppose X is a locally compact space, and $f : X \to Y$ is a continuous surjective map. If f is an open map, then show that Y is locally compact. Give an example of a continuous image of a locally compact space, which fails to be locally compact.

Hint : Consider $\mathbb{Q}$ with discrete topology and the usual topology.

$$8 + 2 = 10$$

6) Given a collection of (nonempty) spaces $\{X_\alpha\}$, consider the product $X = \Pi_\alpha X_\alpha$. Show that the following are equivalent.

a) X is locally compact.

b) Each X_α is locally compact, and moreover X_α is compact for all but finitely many α.

$5 + 5 = 10$

7) Suppose X is a noncompact space, and $\iota : X \hookrightarrow \hat{X}$ is a compactification with $\left|\hat{X} \setminus \iota(X)\right| = 1$. If $\hat{X}$ is T_2, then show that $\hat{X}$ is homeomorphic to the Alexandroff compactification $X^\star$ of X.

10

8) Let X, Y, Z be noncompact spaces, with their Alexandroff compactifications $\hat{X}, \hat{Y}, \hat{Z}$.

a) Given a set map $f : X \to Y$, one can define the set map

$$\begin{aligned} \hat{f} : \hat{X} &\longrightarrow \hat{Y} \\ x &\longmapsto f(x) \\ \infty_X &\longmapsto \infty_Y \end{aligned}$$

Check that $\widehat{g \circ f} = \hat{g} \circ \hat{f}$ for set maps $f : X \to Y, g : Y \to Z$, and also $\widehat{\mathrm{Id}_X} = \mathrm{Id}_{\hat{X}}$.

b) Prove that $\hat{f}$ is continuous if and only if $f : X \to Y$ is continuous and proper (i.e, for any compact set $K \subset Y$, the pre-image $f^{-1}(K)$ is compact).

c) Show that a set map $f : X \to Y$ is a homeomorphism, if and only if $\hat{f} : \hat{X} \to \hat{Y}$ is a homeomorphism.

$3 + 3 + 4 = 10$