Assignment 5

Topology (KSM1C03)

Submission Deadline: 5th October, 2025

1) Show the following implications.

   a) Sequentially compact $\Rightarrow$ countably compact.

   b) Countably compact $\Rightarrow$ limit point compact.

$6 + 4 = 10$

2) Show that the lower limit topology is first countable, but not second countable.

   Hint : For each $x \in \mathbb{R}$, consider $x \in [x, \infty)$, and get basic open sets to derive contradiction.

$4 + 6 = 10$

3) Given any compact space X, show that the cone CX, and the suspension ΣX are compact. Justify that the n-shpere $\mathbb{S}^n$ and the n-disc $\mathbb{D}^n$ are compact.

$4 + 4 + 2 = 10$

4) Suppose $\mathcal{F} \subset \mathcal{P}(X)$ is a family of subsets, such that $\emptyset \notin \mathcal{F}$. Suppose $\mathcal{F}$ has the finite intersection property (FIP) : for any $A_1, \ldots, A_n \in \mathcal{F}$, we have $\bigcap_{i=1}^{n} A_i \neq \emptyset$. Construct a filter $\mathfrak{F}$ containing $\mathcal{F}$. Show that $\mathcal{F}$ is contained in an ultrafilter $\mathfrak{U}$.

$3 + 2 = 5$

5) A filter $\mathcal{F}$ is called *maximal* if $\mathcal{F} \subset \mathcal{G}$ for any filter $\mathcal{G}$ implies $\mathcal{F} = \mathcal{G}$. Show that a filter is an ultrafilter if and only if it is a maximal filter.

   Hint : If $S \notin \mathcal{F}, X \setminus S \notin \mathcal{F}$, then check that $\mathcal{F} \cup \{S\}$ has FIP.

$5 + 5 = 10$

6) Show that a space X is Hausdorff if and only if every ultrafilter on X converges to at most one point.

   Hint : If X is not Hausdorff, there are points $x \neq y$ such that every open nbd of x intersects every open nbd of y. Consider the collection

$$\mathcal{A} := \{U \mid x \in U, U \text{ is open}\} \cup \{V \mid y \in V, V \text{ is open}\}.$$

   Close it under supersets to get a filter, and then get an ultrafilter containing it.

$5+5=10$

7) Let $f : X \to Y$ be a set map, $\mathfrak{U}$ be a filter on X. Define the *pushforward* as

$$f_*\mathfrak{U} := \left\{ A \subset Y \;\middle|\; f^{-1}(A) \in \mathfrak{U} \right\}.$$

Then, show that $\mathfrak{U}$ is a filter on Y. If $\mathfrak{U}$ is an ultrafilter, then show that $f_*\mathfrak{U}$ is an ultrafilter.

$3+2=5$

8) Show that the product space $[0,1]^{[0,1]}$ is not first countable.

Hint : If $\{U_n\}$ is any countable collection of open sets, verify that there exists some $\alpha \in [0,1]$, for which $\pi_\alpha(U_n) = [0,1]$ for all $n \geq 1$, where $\pi_\alpha : [0,1]^{[0,1]} \to [0,1]$ is the α^{th}-component projection.

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