Assignment 4

Topology (KSM1C03)

Submission Deadline: 5th October, 2025

1) Consider the set $X = (-2,-1) \cup \{0\} \cup (1,2) \subset \mathbb{R}$. Show that with the subspace topology, X is not path connected. Now, equip X with the topology $\mathcal{T}$ generated by the base

$$\mathcal{B} := \{(a,b) \mid -2 < a < b < -1\} \bigcup \{(a,b) \mid 1 < a < b < 2\} \bigcup \{(a,-1) \cup \{0\} \cup (1,b) \mid -2 < a < -1,\ 1 < b < 2\}.$$

Show that $(X, \mathcal{T})$ is homeomorphic to $(-2,2)$ (and hence path connected).

$2 + 8 = 10$

2) A collection $\Sigma \subset \mathcal{P}(X)$ of subsets of X is said to have the *finite intersection property (FIP)* if any finite sub-collection of Σ has nonempty intersection.
Show that a space X is compact if and only if given any collection Σ of closed sets of X with FIP, has nonempty intersection (i.e, $\bigcap_{F \in \Sigma} F \neq \emptyset$).

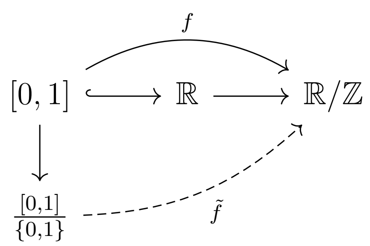
$5 + 5 = 10$

3) Prove the following.

a) Let $X = [0,1]$, any $Y = X/_{\{0,1\}}$ which is the quotient space. Then, Y is homeomorphic to the circle S^1.

b) Let $X = \mathbb{R}$, and $Y = \mathbb{R}/\mathbb{Z}$ be the quotient space obtained by the equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then, Y is homeomorphic to S^1.

Hint : Look at this diagram



where $f(t) = t + \mathbb{Z}$. The induced map $\tilde{f}$ is continuous and bijective. Argue that $\mathbb{R}/\mathbb{Z}$ is T_2. Conclude that $\tilde{f}$ is a homeomorphism.

$3 + 7 = 10$

4) Let X be the real line equipped with the lower limit topology.

a) Show that $[0,1]$ is not compact in X.

b) Let C be compact set in X. Show that C is compact in the usual topology, and hence closed and bounded in X as well. (Recall : lower limit topology is (strictly) finer than the usual topology.)

c) Let C be a subset in X. Suppose $\{x_n\}$ is a strictly increasing sequence in C, i.e, $x_i \in C$ and $x_1 < x_2 < \dots$. Show that C is not compact. (If C is compact, we have a minimum $a = \min C = \inf C$ and a maximum $b = \max C = \sup C$. Consider $U_0 = [a, x_0), U_i = [x_i, x_{i+1}), V = [x_0, \infty)$, where $x_0 = \sup x_i$.)

d) Let $C \subset X$ be a closed set, such that C contains no strictly increasing sequence. Then, show that C is closed in the usual topology as well.

e) Let $C \subset \mathbb{R}$ be a set such that there is no strictly increasing sequence in C. For any $x \in C$, consider the set $S_x := \{y \in C \mid y < x\}$. If $S_x \neq \emptyset$ (i.e, if x is *not* the minimum element of C), denote $s_x := \sup S_x$. If C has a minimum element x, then set $s_x := x - 1$ for convenience.

   i) Show that $s_x \neq x$, and $C \cap (s_x, x) = \emptyset$. In other words, s_x is the previous element of x in C (called *predecessor*).

   ii) Show that $\{I_x := (s_x, x) \mid x \in C\}$ is a collection of disjoint open sets of X.

   iii) Conclude that C is countable.

   In particular, observe that any compact subset of X must be countable.

f) Suppose $C \subset X$ is a closed, bounded subset, without any strictly increasing sequence. Show that C is compact. (Hint: For any cover $\mathcal{U} = \{U_\alpha\}$, start with $x_0 = \inf C \in C$, and some $x_0 \in U_{\alpha_0}$. Next, get $x_1 := \inf(C \setminus U_{\alpha_0})$. Argue that $x_0 < x_1 \in C$. If this process does not terminate after finitely many steps, arrive at a contradiction.)

Observe that $C \subset X$ is compact if and only if C is compact in the usual topology, and contains no strictly increasing sequence.

$$2 + 2 + 2 + 4 + (3 + 3 + 4) + 5 = 25$$

5) Given a space X, the *cone of X* is defined as the quotient space

$$CX := \frac{X \times [0,1]}{X \times \{0\}}.$$

a) Show that CX is homeomorphic to $\frac{X \times [a,b]}{X \times \{a\}}$ and to $\frac{X \times [a,b]}{X \times \{b\}}$ for any interval $[a,b]$.

b) Show that CX is always path connected, but not necessarily locally path connected. (Recall the broom space.)

c) Show that $C\mathbb{S}^n \cong \mathbb{D}^{n+1}$, where $\mathbb{S}^n = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 = 1\right\}$ is the n^{th} sphere, and $\mathbb{D}^{n+1} = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} x_i^2 \leq 1\right\}$ is the $(n+1)^{\text{th}}$ disc.

   Hint : Consider the map

$$\begin{aligned} f : \mathbb{S}^n \times [0,1] &\longrightarrow \mathbb{D}^n \\ (x_1, \dots, x_{n+1}, t) &\longmapsto (tx_1, \dots, tx_{n+1}). \end{aligned}$$

$$(3 + 3) + 4 + 5 = 15$$

6) Prove that $\mathbb{D}^{n+1}/\mathbb{S}^n \cong \mathbb{S}^{n+1}$, where $\mathbb{S}^n \subset \mathbb{D}^{n+1}$ is included as the boundary.

Hint : Consider the map

$$\begin{aligned} f : \mathbb{D}^{n+1} &\longrightarrow \mathbb{S}^{n+1} \\ \mathbf{x} &\longmapsto \left(2\sqrt{1-\|\mathbf{x}\|^2}\mathbf{x}, 2\|\mathbf{x}\|^2 - 1\right), \end{aligned}$$

where $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$, for $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$.

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7) On the space $X \times [0,1]$, consider the following equivalence relation : for any $(x,s),(y,t) \in X \times [0,1]$, define $(x,s) \sim (y,t)$ if and only if one of the following holds

$$s = t \in (0,1), \text{ and } x = y, \qquad \text{or} \qquad s = t = 0, \qquad \text{or} \qquad s = t = 1.$$

The *suspension of X* is defined as the quotient space of $\Sigma X := (X \times [0,1])/_\sim$.

a) Show that ΣX is independent of the choice of the interval. That is, for some $a < b$, if we consider $\Sigma' X = X \times [a,b]/_{\sim'}$, where $(x,s) \sim' (y,t)$ if and only if

$$s = t \in (a,b), \text{ and } x = y, \qquad \text{or} \qquad s = t = a, \qquad \text{or} \qquad s = t = b,$$

then ΣX is homeomorphic to $\Sigma' X$.

b) Show that ΣX is homeomorphic to $\frac{CX}{X \times \{1\}}$, where $X \times \{1\}$ is included in $CX = \frac{X \times [0,1]}{X \times \{0\}}$.

c) Show that $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$.

$3 + 3 + 4 = 10$