Assignment 3

Topology (KSM1C03)

Submission Deadline: 5th October, 2025

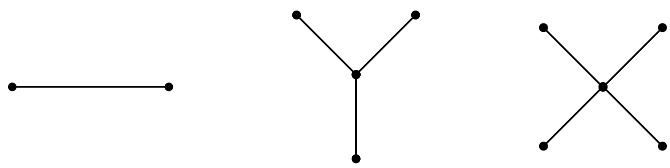
1) Given a space X, define a relation

$$a \sim b \Leftrightarrow a \text{ and } b \text{ are in the same connected component.}$$

   a) Check that $\sim$ is an equivalence relation.
   b) Prove that the connected componetns of X are disjoint closed sets, whose union is X.
   c) Given an example where the connected components are not open.
   d) If X only has finitely many components, show that the quotient space $Y = X/_\sim$ is a discrete space.

$3 + 3 + 2 + 2 = 10$

2) a) Suppose X and Y are homeomorphic. Show that there is an induced bijection between the connected components of X and Y.
   b) Conclude that none of the following shapes (as subspaces of $\mathbb{R}^2$) are homeomorphic to each other.



(i) (ii) (iii)

Hint : If $f : X \to Y$ is a homeomorphism, then we have an induced homeomorphism $\tilde{f} : X \setminus \{x\} \to Y \setminus \{f(x)\}$ for any $x \in X$.

   c) Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$.
   Note : similar argument can be used to show that the circle $\mathbb{S}^1$ is not homeomorphic to the sphere $\mathbb{S}^2$ (or any other $\mathbb{S}^n$ for $n \geq 2$).
   d) Why does this argument cannot be used to show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$?

$5 + 3 \times 3 + 4 + 2 = 20$

3) Suppose $\{A_\alpha \subset X\}_{\alpha \in I}$ is a collection of connected subsets of X. If $\cap A_\alpha \neq \emptyset$, then show that $\cup A_\alpha$ is a connected set. Given an example when A, B are connected but $A \cup B$ is not connected.

$8 + 2 = 10$

4) Prove that the following spaces are totally disconnected.

   a) $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$.

   b) $\left\{\frac{1}{n}\right\} \cup \{0\}$ as a subspace of $\mathbb{R}$

   c) The Sorgenfrey line $\mathbb{R}_l$ (i.e, $\mathbb{R}$ with the lower limit topology).

$5 \times 3 = 15$

5) Recall the K-topology $\mathbb{R}_K$ on $\mathbb{R}$ given by the basis

$$\mathcal{B} := \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) \setminus K \mid a,b \in \mathbb{R}\},$$

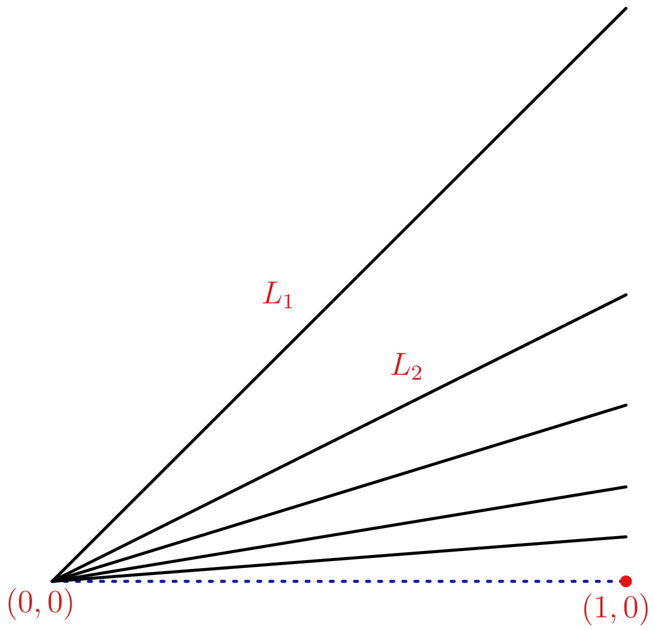
where $K = \left\{\frac{1}{n} \mid n \geq 1\right\}$.

   a) Show that the inclusions $(0,\infty) \hookrightarrow \mathbb{R}_K$, and $(-\infty,0) \hookrightarrow \mathbb{R}_K$ are homeomorphisms onto the image.

   b) Conclude that $\mathbb{R}_K$ is connected.

$5 + 5 = 10$

6) For each $n \geq 1$, consider the line L_n joining $(0,0)$ to $\left(1,\frac{1}{n}\right)$ in $\mathbb{R}^2$. Finally, denote L_0 to be the line joining $(0,0)$ to $(0,1)$. The *broom space* is defined to be the union $\bigcup_{n\geq 0} L_n$ as a subspace of $\mathbb{R}^2$.



Broom space. Removing $(0,1)\times\{0\}$, we get the deleted broom space.

The *deleted broom space* is defined by removing the open segment $(0,1)\times\{0\}$ from the broom space.

Prove that the deleted broom space is connected, but not path connected.

Hint : Use the gradient function $m(x,y) = \frac{y}{x}$, which is a well-defined continuous function away from the y-axis. Note that after removing the origin, m maps the broom space to the totally disconnected space $\{0\} \cup \left\{\frac{1}{n} \mid n \geq 1\right\}$.

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7) Give examples of the following cases, with justification.

a) X is both connected and locally connected.

b) X is not connected, but locally connected.

c) X is connected, but not locally connected. (Hint: Think of the broom space or the comb space)

d) X is neither connected nor locally connected.

$$2\frac{1}{2} \times 4 = 10$$