Assignment 2

Topology (KSM1C03)

Submission Deadline: 5th October, 2025

1) Let X_α be a family of spaces, and $A_\alpha \subset X_\alpha$. Consider $A := \Pi A_\alpha \subset X := \Pi X_\alpha$. Show that $\overline{A} = \Pi \overline{A_\alpha}$, for both product and box topology on X.

   $5 + 5 = 10$

2) Given an infinite collection $\{X_\alpha\}_{\alpha \in I}$ of (non-empty) spaces, consider the product space $X = \Pi X_\alpha$. Fix a point $z = (z_\alpha) \in X$. Consider the subset

   $$A := \{(x_\alpha) \in X \mid x_\alpha = z_\alpha \text{ for all but finitely many } \alpha \in I\}.$$

   Show that $\bar{A} = X$.

   Prove (or disprove by example) the same if X is given the box topology.

   $7 + 3 = 10$

3) Let X be an infinite set, equipped with the cofinite topology. Prove the following.

   a) X is T_1 but not T_2.

   b) A sequence $\{x_n\}$ in X which is eventually constant, say, $x_n = x$ for some $n \geq N$, converges to only x and no other point.

   c) A sequence $\{x_n\}$ in X where no point is repeated infinitely many times, converges to every point in X.

   $3 + 3 + 4 = 10$

4) Let X be an uncountable set, equipped with the cocountable topology. Prove the following.

   a) X is T_1 but not T_2.

   b) If a sequence $\{x_n\}$ in X converges to some x, the sequence is eventually constantly equal to x, i.e., there is some $N \geq 1$ such that $x_n = x$ for all $n \geq N$.

   c) Any sequence in X can converge to at most one point.

   $3 + 3 + 4 = 10$

5) Given a collection of T_2-spaces (even T_1 will suffice) $\{X_\alpha\}$, denote the product set $X = \Pi X_\alpha$. Suppose $x_n = (x_{n,\alpha})$ is a sequence of points, and let $x = (x_\alpha) \in X$. Prove the following.

   a) $x_n \to x$ in the product topology, if and only if $x_{n,\alpha} \to x_\alpha$ in X_α for each α.
   (Note : you don't need to assume T_2 for this case.)

b) $x_n \to x$ in the box topology, if and only if

i) $x_{n,\alpha} \to x_\alpha$ in each X_α, and moreover,

ii) there is a finite set of $S = \{\alpha_1, \dots, \alpha_n\}$ of indices, and an integer $N \geq 1$, such that $x_{n,\alpha} = x_\alpha$ for all $n \geq N$ and for all $\alpha \notin S$.

That is to say, $x_n \to x$ in the box topology if and only if it converges component-wise, and moreover, all but finitely many components are uniformly eventually constant.

(Note : you need to assume T_2 for the "only if" part.)

$4 + 6 = 10$

6) Let $X = \mathbb{R}/\mathbb{Q}$ be the quotient space induced by the relation $x \sim y$ if and only $x - y \in \mathbb{Q}$. Prove that X has the indiscrete topology.

5

7) a) Suppose $A \subset X$ is a subspace and $f : A \to Y$ is a continuous map. On the disjoint union $X \sqcup Y$, consider the relation $u \sim v$ if and only if

i) $u = v$, or

ii) $u, v \in A,\ f(u) = f(v)$, or

iii) $u \in A,\ v = f(u)$, or

iv) $v \in A,\ u = f(v)$, or

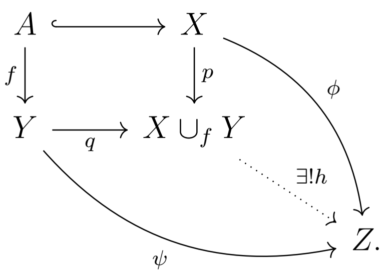
Check that $\sim$ is an equivalence relation on $X \sqcup Y$.

b) Let us denote the quotient space under the equivalence relation as $X \cup_f Y$, which is called the *attaching space* obtained by the *attaching map* f. Check that the maps

$$\begin{aligned} p : X &\to X \cup_f Y \\ x &\mapsto [x], \end{aligned} \quad \text{and} \quad \begin{aligned} q : Y &\to X \cup_f Y \\ y &\mapsto [y] \end{aligned}$$

are continuous. Moreover, check that $p|_A = q \circ f$.

c) Suppose we have maps $\phi : X \to Z$ and $\psi : Y \to Z$ such that the outer square in the diagram commutes (i.e, $\phi|_A = \psi \circ f$):



Then, show that there exists a unique continuous map $h : X \cup_f Y \to Z$ making the triangles commutative, i.e, $h \circ p = \phi$ and $h \circ q = \psi$.

$5 + (3 + 2) + 5 = 15$

8) Suppose $A \subset X$ is a subspace, and consider the identification space X/A (which identifies the points of A to each other, and any point outside A to itself). Say, $q : X \to X/A$ is the identification map.

a) Suppose $f : X \to Z$ is a continuous map, such that $f|_A$ is constant. Then, show that there exists a unique continuous map $\tilde{f} : X/A \to Z$ such that $\tilde{f} \circ q = f$.

b) Consider the constant map $f : A \to Y = \{\star\}$ (i.e, single-point space). Prove that the attaching space $X \cup_f Y$ is homeomorphic to the identification space X/A.

$2 + 8 = 10$

9) Suppose $\mathcal{F} = \{f_\alpha : (X_\alpha, \mathcal{T}_\alpha) \to Y\}$ is a given family of maps. Consider the collection

$$\mathcal{T} \coloneqq \left\{U \subset Y \;\middle|\; f_\alpha^{-1}(U) \in \mathcal{T}_\alpha \;\forall\, f_\alpha\right\}.$$

Prove that $\mathcal{T}$ is a topology (called the topology *co-induced by the family* $\mathcal{F}$).

Prove that any map $f : (Y, \mathcal{T}) \to (Z, \mathcal{T}_Z)$ is continuous if and only if $f \circ f_\alpha : (X_\alpha, \mathcal{T}_\alpha) \to (Z, \mathcal{T}_Z)$ is continuous for all α.

$5 + 5 = 10$