Assignment 10

Topology (KSM1C03)

Submission Deadline: 21th November, 2025

1) **(Sum of two metric)** Suppose d_1, d_2 are two metric on X, inducing the topologies $\mathcal{T}_1, \mathcal{T}_2$ respectively.

   a) Check that $d = d_1 + d_2$ is a metric on X. Denote the topology by $\mathcal{T}$.

   b) Show that $\mathcal{T}$ is finer than $\mathcal{T}_1$ (and symmetrically, than $\mathcal{T}_2$).

   c) If $\mathcal{T}_1 = \mathcal{T}_2$, show that $\mathcal{T} = \mathcal{T}_1 = \mathcal{T}_2$.

$2 + 3 + 5 = 10$

2) **(Uniform metric)** Suppose (X_α, d_α) is a collection of (nonempty) metric spaces. Denote the bounded metric $\bar{d}_\alpha(x, y) = \min\{d_\alpha(x, y), 1\}$ for $x, y \in X_\alpha$. On the product $X = \prod X_\alpha$, consider

$$d(x, y) := \sup_\alpha \left\{\bar{d}_\alpha(x_\alpha, y_\alpha)\right\}, \qquad x = (x_\alpha),\ y = (y_\alpha) \in X.$$

   a) Check that d is a metric on X.

   b) If each d_α is complete, show that d is complete.

   c) Conversely, if d is complete, show that each d_α is complete.

   The induced metric topology, known as the *uniform topology*, need not be the product topology, as we have seen that the product space $\mathbb{R}^{[0,1]}$ is not even first countable, and hence, not metrizable.

$2 + 4 + 4 = 10$

3) **(Isometry is embedding)** Let $f : (X, d_X) \to (Y, d_Y)$ be an isometry.

   a) Show that f is an embedding, i.e, f is injective, continuous, and open onto the image. In other words, f is a homeomorphism of X and the subspace $f(X) \subset Y$.

   b) Show that f takes Cauchy sequence to Cauchy sequence.

   c) Suppose f is surjective.

      i) Show that the inverse $f^{-1} : (Y, d_Y) \to (X, d_X)$ is an isometry, and hence, f is a homeomorphism.

      ii) Show that d_X is complete if and only if d_Y is complete.

$3 + 3 + 2 + 2 = 10$

4) Let $A \subset (X, d)$ be a dense subset. Suppose every Cauchy sequence in $(A, d|_A)$ converges in X, where $d|_A$ is the restricted metric. Show that (X, d) is complete.

10

5) Suppose X is a T_3-space.

   a) Suppose $A \subset X$ is a closed set. Show that the identification space $Y = X/A$ is T_2.

   b) Suppose $f : X \to Y$ is a surjective, open, closed, continuous map. Show that Y is T_2.

      Hint : Note that f is a quotient map. Show that the set $\Delta = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$ is closed in $X \times X$.

$4 + 6 = 10$

6) **(Closed continuous image of T_3-space)** Recall the Moore plane $X = H \cup L$, where $H = \{(x, y) \in \mathbb{R} \mid y > 0\}$ and $L = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$. The topology on H is the usual one, and for any point $(x, 0) \in L$ a basic open neighborhood is of the form $\{(x, 0)\} \cup D$, where $D \subset H$ is an open disc tangentially touching L at $(x, 0)$. We have seen X is $T_{3\frac{1}{2}}$ but not T_4. Let us explicitly produce two disjoint closed sets, that cannot be separated by open sets.

   Consider the sets $Q = \{(x, 0) \mid x \in \mathbb{Q}\}$ and $I = \{(x, 0) \mid x \in \mathbb{R} \setminus \mathbb{Q}\}$. As L is a closed set with the discrete topology, we have Q and I are disjoint closed sets in X.

   a) Show that Q and I cannot be separated by disjoint open sets in T_4.

      Hint : Say, $Q \subset U, I \subset V$. For each $x \in I$, consider tangent disc D_x of radius r_x. Then, we have a countable cover of $\mathbb{L}$ by $Q \cup \bigcup_{n \geq 1} \left\{x \in I \mid r_x > \frac{1}{n}\right\}$. Use the Baire category theorem to get an interval (a, b) and some n_0 such that $\left\{x \in (a, b) \cap I \,\middle|\, r_x > \frac{1}{n_0}\right\}$ is dense in (a, b) (in the usual topology). Argue that any basic open neighborhood of some $x \in (a, b) \cap Q$ in X must intersect V.

   b) Consider the identification space $Y = X/Q$.

      i) Verify that the quotient map $q : X \to Y$ is a closed map.

      ii) Show that Y is T_2, but not T_3. Thus, a closed, continuous image of a $T_{3\frac{1}{2}}$-space may fail to be $T_{3\frac{1}{2}}$.

$6 + (2 + 2) = 10$

7) **(Banach fixed-point theorem)** Let $f : (X, d) \to (X, d)$ be a function of a complete metric space. Suppose, for some $0 < \rho < 1$ we have

$$d\left(f(x), f(y)\right) \leq \rho d\left(x, y\right), \quad x, y \in X.$$

   a) Show that f is continuous.

   b) Show that f has a unique fixed point, i.e, there is a unique $x_0 \in X$ satisfying $f(x_0) = x_0$.

$4 + 6 = 10$