Assignment 1

Topology (KSM1C03)

Submission Deadline: 19^{th} August, 2024

1) Recall the equivalence relation which identifies a subset $A \subset X$ to a singleton.
   a) Identify the equivalence classes X/A, when $X = [0,1]$ (the closed interval), and $A = \{0,1\}$ (the endpoints).
   b) Identify the equivalence classes X/A when $X = \mathbb{R}$ (the real line), and $A = \mathbb{Z}$ (the integers).
   c) Can you see a natural bijection between $[0,1]/\{0,1\}$ and $\mathbb{R}/\mathbb{Z}$?

$3+3+4=10$

2) On any set X, consider the following collections of subsets.
   a) $\mathcal{T}_1 := \{A \subset X \mid X \setminus A \text{ is finite}\} \bigcup \{\emptyset\}$.
   b) $\mathcal{T}_2 := \{A \subset X \mid X \setminus A \text{ is countable}\} \bigcup \{\emptyset\}$.

   Show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, respectively called the *cofinite* and the *cocountable* topologies.
   Now, suppose X is uncountable (say, $X = \mathbb{R}$), and consider the collection

$$\mathcal{T}_3 := \{A \subset X \mid X \setminus A \text{ is uncountable}\}.$$

   Is $\mathcal{T}_3$ a topology on X?

$4+4+2=10$

3) On the real line $\mathbb{R}$, consider the collection of all half-open intervals

$$\mathcal{B}_l := \{[a,b) \mid a, b \in \mathbb{R}\}.$$

   Show that $\mathcal{B}_l$ is a basis for a topology on $\mathbb{R}$ (called the *lower limit topology*). The real line equipped with the lower limit topology is also known as the *long line* or the *Sorgenfrey line*.

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4) Suppose $(X, \mathcal{T})$ is a topological space, and $\mathcal{S} \subset \mathcal{T}$ is a sub-collection. Prove that the following are equivalent.
   a) $\mathcal{S}$ is a subbasis of $\mathcal{T}$.
   b) $\mathcal{T}$ is the intersection of all possible topologies on X, that contains $\mathcal{S}$.
   c) The collection $\mathcal{B}$ of all possible finite intersections of elements of $\mathcal{S}$ is a basis for $\mathcal{T}$.

$5+5+5=15$

5) Denote, $K = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} \subset \mathbb{R}$. Consider the collection,

$$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}\} \bigcup \{(a,b) \setminus K \mid a,b \in \mathbb{R}\}.$$

Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$ (called the K-topology). Prove that the K-topology is strictly finer than the usual topology on $\mathbb{R}$, but is not comparable to the lower limit topology.

$5 + 3 + 2 = 10$

6) Let X be a space.

a) For $A, B \subset X$, if $A \subset B$ then $\bar{A} \subset \bar{B}$. Give an example where $A \subsetneq B$ but $\bar{A} = \bar{B}$.

b) Given a finite collection $A_1, \dots, A_n \subset X$, show that

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}.$$

c) Given any infinite collection of subsets $A_\alpha \subset X$ (for $\alpha \in \mathcal{I}$, some indexing set), show that

$$\overline{\bigcup_{\alpha \in \mathcal{I}} A_\alpha} \supset \bigcup_{\alpha \in \mathcal{I}} \overline{A_\alpha}.$$

Give an example of a countably infinite collection, where equality doesn't hold.

d) Given any collection of subsets $A_\alpha \subset X$ (for $\alpha \in \mathcal{I}$, some indexing set), show that

$$\overline{\bigcap_{\alpha \in \mathcal{I}} A_\alpha} \subset \bigcap_{\alpha \in \mathcal{I}} \overline{A_\alpha}.$$

Give an example of $A, B \subset X$ such that $\overline{A \cap B} \subsetneq \bar{A} \cap \bar{B}$.

$(4 + 1) + 5 + (4 + 1) + 5 = 20$

7) Compute the boundary of the following subsets $A \subset X$.

a) X is any space, and $A = X$.

b) X is any space, and $A = \emptyset$.

c) X is a discrete space, and $\emptyset \neq A \subsetneq X$.

d) X is an indiscrete space, and $\emptyset \neq A \subsetneq X$.

e) $X = \mathbb{R}$ and $A = \mathbb{Z}$.

f) $X = \mathbb{R}$ and $A = \mathbb{Q}$.

g) $X = \mathbb{R}$ and $A = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$.

$1 \times 4 + 2 \times 3 = 10$

8) Suppose $(X, \mathcal{T})$ is a topological space. Show that the following are equivalent.

a) X has the discrete topology, i.e., $\mathcal{T} = \mathcal{P}(X)$.

b) Given any space Y, any function $f : X \to Y$ is continuous.

c) The map $\mathrm{Id} : (X, \mathcal{T}) \to (X, \mathcal{P}(X))$ is continuous.

$5 + 5 + 5 = 15$

9) Suppose $(X, \mathcal{T})$ is a space, and some $A \subset X$ is equipped with the subspace topology $\mathcal{T}_A$.

a) Show that the inclusion map $\iota : A \hookrightarrow X$ is continuous.

b) Suppose $\mathcal{S}$ is some topology on A such that the inclusion map $\iota : (A, \mathcal{S}) \hookrightarrow (X, \mathcal{T})$ is continuous. Show that $\mathcal{S}$ is finer than $\mathcal{T}_A$.

$5 + 5 = 10$

10) Let X be a set and $\mathcal{F} = \{f_\alpha : X \to Y_\alpha\}_{\alpha \in \mathcal{I}}$ be a collection of functions, where $(Y_\alpha, \mathcal{T}_\alpha)$ is a topological space. Consider the collection

$$\mathcal{S} \coloneqq \left\{ f^{-1}(U) \;\middle|\; f = f_\alpha \in \mathcal{F},\ U \in \mathcal{T}_\alpha \right\}.$$

Show that the topology on X generated by $\mathcal{S}$ (as a subbasis) is the smallest (i.e., coarsest) possible topology on X such that each $f_\alpha : (X, \mathcal{T}) \to (Y_\alpha, \mathcal{T}_\alpha)$ is continuous.

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