Topology Course Notes (KSM1C03)

Day 30 : 20th November, 2025

`paracompactness -- partition of unity`

30.1 Paracompactness (Cont.)

Proposition 30.1

uppose $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X, which admits a locally finite refinement $\mathcal{V} = \{V_j\}_{j \in J}$. Then, there exists a locally finite refinement $\mathcal{W} = \{W_i\}_{i \in I}$ such that $W_i \subset U_i$ for all $i \in I$.

Proof

Suppose $\phi : J \to I$ is the function such that $V_j \subset U_{\phi(j)}$ for each $j \in J$. For each $i \in I$, consider the set

$$W_i := \bigcup \{V_j \mid \phi(j) = i\} = \bigcup_{j \in \phi^{-1}(i)} V_j.$$

Clearly, $W_i \subset V_i$ for all $i \in I$, and $\mathcal{W} = \{W_i\}_{i \in I}$ still covers X. Thus, $\mathcal{W}$ is a refinement of $\mathcal{U}$ (but now with same indexing). We need to show that $\mathcal{W}$ is locally finite. Let $x \in X$ be fixed. Then, there is an open neighborhood $x \in N \subset X$, such that $N \cap V_j = \emptyset$ for all $j \in J \setminus A$, where $A \subset J$ is a finite set. Then, $B = \phi(A) \subset I$ is also a finite set. If possible, for some $i \in I \setminus B$, suppose $N \cap W_i \neq \emptyset$. Then, $N \cap \left(\bigcup_{\phi(j)=i} V_j \right) \neq \emptyset$. So, for some $j \in J$ with $\phi(j) = i$, we must have $N \cap V_j \neq \emptyset$. But then we must have $j \in B \Rightarrow i = \phi(j) \in \phi(B) = A$, a contradiction. Hence, $N \cap W_i = \emptyset$ for all $i \in I \setminus B$. Thus, $\mathcal{W}$ is a locally finite refinement of $\mathcal{U}$. $\square$

Example 30.2: (Compact and Lindelöf space)

Since for a compact space, you can get a finite sub-cover of any open cover, it will clearly be a locally finite refinement. Thus, any compact space is paracompact. A Lindelöf space may not be paracompact! As an example, consider the double-origin plane. We have seen that it is $T_{2\frac{1}{2}}$ but not T_3. Also, it is second countable, and hence, Lindelöf. On the other hand, it cannot be paracompact, as a paracompact T_2 space is T_4.

Proposition 30.3: (Closed subset of Paracompact)

Let X be a paracompact space, and $C \subset X$ be closed. Then, C is paracompact.

Proof

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of C. Suppose $U_i = C \cap \tilde{U}_i$, where $\tilde{U}_i \subset X$ is open. Then, $\widetilde{\mathcal{U}} = \{X \setminus C\} \cup \left\{\tilde{U}_i\right\}_{i\in I}$ is an open cover of X. By paracompactness, we have a locally finite refinement, say, $\mathcal{V} = \{V_0\} \cup \{V_i\}_{i\in I}$, so that $V_0 \subset X \setminus C$ and $V_i \subset \tilde{U}_i$ for all $i \in I$. Now, for any $x \in C$, there is some open set $x \in N \subset X$ such that $N \cap V_i = \emptyset$ for all $i \in I_0 \setminus F$, where $F \subset I_0 := I \sqcup \{0\}$ is a finite subset. Then, clearly $N \cap X \cap V_i = \emptyset$ for any $i \in I \setminus F$. Thus, $\{V_i \cap C\}_{i\in I}$ is a locally finite refinement of $\mathcal{U}$. Consequently, C is paracompact. □

Theorem 30.4: (Paracompact T_2 is T_4)

A paracompact T_2 space is T_4.

Proof

Let X be a paracompact T_2 space. Let us first proof regularity of X. Say, $A \subset X$ is closed, and $x \in X \setminus A$ is a point. As X is T_2, for each $a \in A$ there are open sets U_a, V_a such that $x \in U_a, a \in V_a$ and $U_a \cap V_a = \emptyset$. Now, $\mathcal{V} = \{X \setminus A\} \cup \{V_a\}_{a\in A}$ is an open cover of X, and hence, there is a locally finite refinement, say, $\mathcal{W}$. Define

$$V := \bigcup \{W \in \mathcal{W} \mid W \cap A \neq \emptyset\}.$$

Note that $A \subset V$. Since $\mathcal{W}$ is a locally finite collection (and hence, so is any subcollection of $\mathcal{W}$), we also have

$$\bar{V} = \bigcup \left\{\bar{W} \mid W \in \mathcal{W},\ W \cap A \neq \emptyset\right\}.$$

Now, any $W \in \mathcal{W}$ with $W \cap A \neq \emptyset$ is contained in some V_a for some $a \in A$, and hence, $\bar{W} \subset \bar{V}_a$. Thus,

$$\bar{V} \subset \bigcup_{a\in A} \bar{V}_a.$$

As $a \in U_a$ and $U_a \cap V_a = \emptyset$, we have $a \notin \bar{V}_a$, and hence, $a \not\bar{V}$. Then, consider $U = X \setminus \bar{V}$. Clearly, $x \in U, A \subset V$ and $U \cap V = \emptyset$. Thus, X is a regular space.

Now, consider $A, B \subset X$ be closed sets, with $A \cap B = \emptyset$. For each $a \in A$, there are open sets $U_a, V_a \subset X$ such that $a \in U_a, B \subset V_a$ and $U_a \cap V_a = \emptyset$. In particular, $B \cap \bar{U}_a = \emptyset$. Again, consider the open cover $\{X \setminus A\} \cup \{U_a\}_{a\in A}$ of X, and get a locally finite refinement, say, $\mathcal{G}$. Define $U = \bigcup \{G \in \mathcal{G} \mid G \cap A \neq \emptyset\}$. Then, $\bar{U} = \bigcup \left\{\bar{G} \mid G \in \mathcal{G},\ G \cap A \neq \emptyset\right\}$ follows from local finiteness. Observe that $B \cap \bar{U} = \emptyset$. Then, set $V = X \setminus \bar{U}$. Clearly, $A \subset U, B \subset V$ and $U \cap V = \emptyset$. Thus, X is normal. As X is T_2, we have X is T_4. □

Example 30.5: ($T_4 \not\Rightarrow$ Paracompact)

Consider $[0, \Omega)$, the first uncountable ordinal with the order topology. We have seen that X is T_4 (in fact, T_5). Now, the product space $[0, \Omega) \times [0, \Omega]$ was shown to be not T_4. If $[0, \Omega)$ was paracompact, then the product with the compact space must be paracompact again. But then the product being paracompact and T_2 has to be normal, a contradiction. Hence, $[0, \Omega)$ is not paracompact. On the other hand, a Lindelöf, regular space is paracompact.

Exercise 30.6: (Product of Compact and Paracompact)

Show that the product of a compact and a paracompact space is again paracompact.

Hint

Use the tube lemma.

30.2 Partition of Unity

Definition 30.7: (Support)

Given a continuous map $f : X \to \mathbb{R}$, the *support of f* is defined as

$$\operatorname{supp}(f) := \overline{f^{-1}(\mathbb{R} \setminus \{0\})}.$$

In words, support is the smalles closed set containing the non-zero set of f.

Definition 30.8: (Partition of Unity)

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. A *partition of unity subordinate to $\mathcal{U}$* is a collection of continuous maps $\{f_i : X \to [0, 1]\}_{i \in I}$ such that the following holds.

i) For each $i \in I$, we have $\operatorname{supp}(f_i) \subset U_i$.

ii) The collection $\{\operatorname{supp}(f_i)\}_{i \in I}$ is a locally finite cover of X.

iii) For each $x \in X$, we have $\sum_{i \in I} f_i(x) = 1$.

The arbitrary sum in the third condition is actually a finite sum by local finiteness.

Theorem 30.9: (Shrinking Lemma)

Let X be a paracompact T_2 space. Then, for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$, there exist a locally finite open cover $\mathcal{V} = \{V_i\}_{i \in I}$ such that $V_i \subset \overline{V_i} \subset U_i$ for all $i \in I$.

Proof

Note that X is T_4 and in particular regular. Consider $\mathcal{W}$ to be the collection of open sets $W \subset X$ such that $W \subset \bar{W} \subset U_i$ for some $i \in I$. As $\mathcal{U}$ is a cover, by regularity, it follows that $\mathcal{W}$ is also a cover. Let us index it as $\mathcal{W} = \{W_j\}_{j \in J}$. We have function (by axiom of choice)

$$\theta : J \to I$$

such that $W_j \subset \overline{W_j} \subset U_{\theta(j)}$ for all $j \in J$. For each $i \in I$, denote

$$V_i = \bigcup \{W_j \mid \theta(j) = i\}.$$

Note that if $\theta^{-1}(i)$ is empty, then $V_i = \emptyset$. Consider the collection $\mathcal{V} = \{V_i\}_{i \in I}$, which is still a cover, and by construction, $V_i \subset U_i$ for all $i \in I$. Now, by local finiteness of $\mathcal{W}$, it follows that $\overline{V_i} = \bigcup_{\theta(j)=i} \overline{W_j} \subset U_i$ as well. Finally, let us check local finiteness of $\mathcal{V}$. For $x \in X$, there is an open set $N \subset X$ such that $N \cap W_j = \emptyset$ for all $j \in J \setminus F$, where $F \subset J$ is a finite set. Then, $\theta(F) \subset I$ is a finite set. Suppose, for some $i \in I \setminus \theta(F)$, we have $N \cap V_i \neq \emptyset$. Then, $N \cap W_j \neq \emptyset$

for some $j \in J$ such that $\theta(j) = i$. But then $j \in F \Rightarrow i = \theta(j) \in \theta(F)$, a contradiction. Thus, $\mathcal{W}$ is a locally finite collection. □

Theorem 30.10: (Existence of Partition of Unity)

In a paracompact T_2 space, any open cover admits a partition of unity subordinate to the cover.

Proof

Let X be a paracompact T_2 space, and $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover. Applying the shrinking lemma twice, we get two open covers $\mathcal{V} = \{V_i\}_{i \in I}$ and $\mathcal{W} = \{W_i\}_{i \in I}$ such that

$$V_i \subset \overline{V_i} \subset W_i \subset \overline{W_i} \subset U_i, \qquad i \in I.$$

Note that $\overline{V_i}$ and $X \setminus W_i$ are disjoint closed sets. Then, by the Urysohn lemma, there are continuous functions $h_i : X \to [0,1]$ such that

$$h_i(\overline{V}_i) = 1, \quad h_i\left(X \setminus W_i\right) = 0.$$

Observe that

$$h_i^{-1}(0,1] \subset W_i \Rightarrow \operatorname{supp}(h_i) = \overline{h_i^{-1}(0,1]} \subset \overline{W_i} \subset U_i.$$

As $\mathcal{V} = \{V_i\}_{i \in I}$ is locally finite, it follows that $\left\{\overline{V_i}\right\}_{i \in I}$ is again a locally finite collection (Check!). Hence, $\{\operatorname{supp}(h_i)\}$ is a locally finite collection. For any $x \in X$, we have $x \in V_i$ for some $i \in I$, and then, $h_i(x) = 1$. Thus, $\{\operatorname{supp}(h_i)\}_{i \in I}$ is a locally finte cover of X. Now, local finiteness implies that $h(x) = \sum_{i \in I} h_i(x)$ is always a finite sum. Let us show that it is in fact a continuous map. Indeed, for any $x \in X$, there is a neighborhood $N \subset X$, such that $h|_N$ is a finite sum of continuous functions, which is then continuous. Since h is continuous on neighborhoods, it follows that h is continuous. Moreover, h is nowhere vanishing (Check!). In fact, we have $h : X \to [0, \infty)$ Define $f_i = \frac{h_i}{h}$, which is again continuous. Note that $f_i : X \to [0,1]$, as $h \geq 1$. Moreover, for each $x \in X$ we have

$$f(x) = \sum f_i(x) = \sum \frac{h_i(x)}{h(x)} = \frac{\sum h_i(x)}{h(x)} = \frac{h(x)}{h(x)} = 1.$$

Clearly, $\operatorname{supp}(f_i) \subset \operatorname{supp}(f_i)$. Thus, $\{f_i : X \to [0,1]\}_{i \in I}$ is partition of unity subordinate to the family $\mathcal{U}$. □