Topology Course Notes (KSM1C03)

Day 29 : 19^{th} November, 2025

`Baire category theorem -- paracompactness`

29.1 Baire Category Theorems

Theorem 29.1: (Baire Category Theorem)

A G_δ-set in a compact T_2 space is a Baire space.

Proof

Let X be compact, T_2-space. Note that X is a T_4-space. Let us first show that X itself is Baire. Let $G_n \subset X$ be a countable collection of open dense sets, and $U \subset X$ be a fixed nonempty open set. Denote $V_0 = U$. Now, $U \cap G_1 \neq \emptyset$. Then, by regularity, there is a nonempty open set V_1, with $\overline{V_1} \subset U \cap G_1$. Inductively, assume that there is a nonempty open set V_n such that $\overline{V_n} \subset V_{n-1} \cap G_n$. Since $V_n \cap G_{n+1} \neq \emptyset$, again by regularity, we have a nonempty open set V_{n+1} with $\overline{V_{n+1}} \subset V_n \cap G_{n+1}$. Now, by construction, $\{\overline{V_n}\}_{n\geq 1}$ are closed sets, with $\overline{V_1} \supset \overline{V_2} \supset \dots$. Consequently, $\{\overline{V_n}\}$ is a collection of (nonempty) closed sets with finite intersection property. Hence, $\bigcap \overline{V_n} \neq \emptyset$. But, $\bigcap \overline{V_n} \subset U \cap \bigcap G_n$ by construction. Thus, $U \cap \bigcap G_n \neq \emptyset$. As U is arbitrary nonempty open set, we have $\bigcap G_n$ is dense in X. Thus, X is a Baire space.

Now, let us consider a G_δ-set $K = \bigcap U_n$, where $U_n \subset X$ is open. Consider $\bar{K}$, which is closed, hence compact, and also T_2. Now, $V_n = U_n \cap \bar{K}$ is an open set in $\bar{K}$. Note that $\bigcap V_n = \bigcap U_n \cap \bar{K} = K \cap \bar{K} = K$. Also, $K \subset V_n \subset \bar{K} \Rightarrow \bar{K} = \overline{V_n}$. Thus, V_n is an open dense set in the compact, T_2 space $\bar{K}$. Now, suppose $W_i \subset K$ are open, dense subsets. Then, $W_i = K \cap G_i$ for some $G_i \subset \bar{K}$ open. Clearly, G_i is also dense in $\bar{K}$, since for any nonempty open set $V \subset \bar{K}$ we have,

$$V \cap K \neq \emptyset \underset{\text{as } W_i \text{ is dense in } K}{\Rightarrow} (V \cap K) \cap W_i \neq \emptyset \Rightarrow V \cap G_i \neq \emptyset.$$

Thus, we have a countable collection $\{G_i\} \cup \{V_n\}$ of open dense subsets in $\bar{K}$. Hence, the intersection

$$\bigcap_i G_i \cap V_i = \left(\bigcap_i G_i\right) \cap \left(\bigcap_i V_i\right) = \left(\bigcap G_i\right) \cap K = \bigcap G_i \cap K = \bigcap W_i$$

is dense in $\bar{K}$. But then $\bigcap W_i$ is dense in K as well. Hence, K is a Baire space. □

Corollary 29.2: (BCT 1)

A locally compact T_2 space is a Baire space.

Proof

Suppose X is locally compact, T_2. A locally compact, T_2 noncompact space embeds as an open subset in its one point compactification $\hat{X}$, which is compact, T_2. Thus, X is a G_δ-set in $\hat{X}$, and hence, a Baire space. □

Theorem 29.3: (BCT 2)

A completely metrizable space is a Baire space

Proof

Let (X, d) be a complete metric space. Suppose $G_i \subset X$ is a countable collection of open dense set, and $U \subset X$ is a fixed nonempty open set. Proceeding as in the proof of Baire category theorem, consider $V_0 = U$, and get open balls $V_n = B_d(x_n, r_n)$ of radius $r_n < \frac{1}{n}$, such that $\overline{V_{n+1}} \subset V_n \cap G_{n+1}$ holds. In particular, we have a decreasing sequence of closed balls $V_0 \supset \overline{V_1} \supset \overline{V_2} \supset \dots$, and moreover, $\bigcap \overline{V_n} \subset U \cap \bigcap G_n$ holds.

We claim that the sequence $\{x_n\}$ is Cauchy. Indeed, for any $\epsilon > 0$, get $N \geq 1$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Then, for any $n, m \geq N$ we have $x_n, x_m \in V_N$. Hence,

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < r_N + r_N < \frac{2}{N} < \epsilon.$$

As X is complete, we have $x_n \to x$. Clearly, $x \in \overline{V_n}$ for all n. Hence, $x \in U \cap G_n$ for all $n \geq 1$. Thus, $U \cap \bigcap_n U_n \neq \emptyset$. As U is arbitrary nonempty open set, we have $\bigcap G_n$ is dense. Thus, X is a Baire space. □

Corollary 29.4: ($\mathbb{Q}$ is not G_δ)

The set of rationals $\mathbb{Q} \subset \mathbb{R}$ is not a G_δ-set.

Proof

If possible, suppose $\mathbb{Q}$ is G_δ. Then, $\mathbb{Q} = \bigcap_n U_n$ for some open sets $U_n \subset \mathbb{R}$. Clearly, U_n is dense in $\mathbb{R}$, since $\mathbb{Q} \subset U_n$ is already dense. Now, foreach $q \in \mathbb{Q}$, consider $V_q = \mathbb{R} \setminus \{q\}$, which are also open and dense. Note that $\bigcap_{q \in \mathbb{Q}} V_q = \mathbb{R} \setminus \mathbb{Q}$. Now, $\{U_n\}_{n \geq 1} \cup \{V_q\}_{q \in \mathbb{Q}}$ is a countable collection of open dense sets. Since $\mathbb{R}$ is a Baire space, there intersection must be dense. But, $\bigcap_{n \geq 1} U_n \cap \bigcap_{q \in \mathbb{Q}} V_q = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, a contradiction. Hence, $\mathbb{Q}$ is not a G_δ-set. □

Remark 29.5

Since for a function $f : \mathbb{R} \to \mathbb{R}$ the set of continuities must be a G_δ-set, it follows that there does not exist a function which is continuous only at the rationals.

Theorem 29.6: (Choquet spaces are Baire space)

Let X be a nonempty space. Then, X is a Choquet space if and only if X is a Baire space.

Proof

Let X be a Choquet space. Suppose G_n is a countable collection of open dense sets. Fix some nonempty open set $O \subset X$. Let player E choose the open set $U_0 := G_1 \cap O$, which is nonempty as G_1 is dense. Suppose at the n^{th}-stage, player N chooses $V_n \subset U_n$ according to their winning strategy. Then, player E chooses $U_{n+1} := V_n \cap G_{n+1}$, which is again nonempty as G_{n+1} is dense. At the end of the game, since N must win, we have

$$\emptyset \neq \bigcap_{n\geq 0} U_n = (O \cap G_1) \cap \bigcap_{n\geq 1} V_n \cap G_{n+1} \subset O \cap \bigcap_{n\geq 1} G_n.$$

As O is an arbitrary nonempty open set, we have $\bigcap G_n$ is dense in X.

Conversely, let X be a Baire space. If possible, suppose player E has a winning strategy,

$$\mathsf{f} : \mathcal{T}_\star \to \mathcal{T}_*,$$

where $\mathcal{T}_\star$ denotes the set of nonempty open sets of X. Say, according to this strategy, player E chooses the open set $U_0 \subset X$. We shall show that U_0 is not a Baire space.

Fix some open $U \subset U_0$. Given any collection $\mathcal{O}$ of nonempty open subsets of U, call $\mathcal{O}$ is *good* if

$$\mathcal{O}^\star = \{\mathsf{f}(O) \mid O \in \mathcal{O}\}$$

is a pairwise disjoint collection of (necessarily nonempty) open subsets of U. Let $\mathfrak{O}_U$ be the collection of all good sub-collections of U, partially ordered by inclusion. For a chain $\{\mathcal{O}_\alpha\}$ in $\mathfrak{O}_U$, consider the union $\mathcal{O} = \bigcup \mathcal{O}_\alpha$. If possible, suppose there are $O_\alpha \in \mathcal{O}_\alpha$ and $O \in \mathcal{O}_\beta$ such that $\mathsf{f}(O_\alpha) \cap \mathsf{f}(O_\beta)$. Without loss of generality, $\mathcal{O}_\alpha \subset \mathcal{O}_\beta$. But as $\mathcal{O}_\beta$ is good, we have a contradiction. Thus, $\mathcal{O}$ is a good sub-collection of nonempty open sets of U. Hence, by Zorn's lemma, we can then get a *maximal* good collection, say, $\mathcal{O}_U^{\mathsf{max}}$. Let us denote

$$U^\star := \bigcup_{O \in \mathcal{O}_U} \mathsf{f}(O).$$

Clearly, $U^\star$ is a nonempty open set of $U^\star$. We claim that $U^\star$ is dense in U. If not, then there is some nonempty open set $O \subset U$ such that $O \cap U^\star = \emptyset$. Then, $\mathsf{f}(O) \subset O$ is a nonempty open set, and clearly, $\mathsf{f}(O) \cap U^\star = \emptyset$. But then, $\mathcal{O}_U^{\mathsf{max}} \cup \{O\}$ is also good, violating the maximality of $\mathcal{O}_U$. Hence, for any $U \subset U_0$, we have constructed $U^\star$, which is open and dense in U_0, and given as the union of pairwise disjoint open sets of the form $\mathsf{F}(O)$ for open subsets $O \subset U$.

Let us now inductively construct the following open dense sets. Set $G_1 = U_0^\star$. Assuming G_n is defined, set $G_{n+1} = \bigcup_{W \in G_n} W^\star$. Observe that each G_n is a *disjoint* union of open sets of the form $\mathsf{f}(U)$ for some open $U \subset U_0$. Moreover, G_{n+1} is dense in G_n, and hence, by a simple induction, each G_n is dense in U_0 as well. If possible, let $x \in \bigcap G_n$. Since $x \in G_1$, we have a unique open $V_0 \subset U_0$, such that $x \in \mathsf{f}(V_0)$ (as G_1 is a disjoint union). Set $U_1 = \mathsf{f}(V_0)$. Inductively, suppose we have constructed $(U_0, V_0, U_1, V_1, \ldots, U_n)$. Now, $x \in G_{n+1}$. Hence, there is a unique open set $V_n \subset U_n$, such that $x \in \mathsf{f}(V_n)$ (as G_{n+1} is a disjoint union). Set $U_n = \mathsf{f}(V_n)$. This is a game of Choquet! Now, by construction, $x \in \bigcap U_n = \bigcap V_n$. Thus, player N wins in this game. This is a contradiction, since player E is playing by a winning strategy by assumption. Hence, we must have $\bigcap G_n = \emptyset$. But then, U_0 is an open set of X, which is not Baire. Consequently, X itself cannot be a Baire space. □

Corollary 29.7: (BCT 1 and 2 by game of Choquet)

X is a Choquet space (and hence, a Baire space) if either a) X is completely metrizable, or b) X locally compact T_2.

Proof

Suppose X completely metrizable. At the n^{th}-stage of any Choquet game, let player N choose $V_n \subset U_n$ satisfying $V_n \subset \overline{V_n} \subset U_n$, and $\mathrm{Diam}\overline{V_n} < \frac{1}{2}\mathrm{Diam}U_n$. Then, a usual argument using Cauchy sequence shows that $\bigcap V_n = \bigcap \overline{V_n} \neq \emptyset$. Thus, X is a Choquet space.

Next, suppose X is a locally compact T_2 space. This time, at the n^{th}-stage, let player N choose $V_n \subset U_n$ satisfying $V_n \subset \overline{V_n} \subset U_n$, and $\overline{V_n}$ compact (this is possible, as the space is locally compact, T_2). It follows that $\bigcap V_i = \bigcap \overline{V_i} \neq \emptyset$, as the intersection of decreasing nonempty closed sets in a compact space (here, the compact space is $\overline{V_1}$) is always nonempty. □

29.2 Paracompactness

Definition 29.8: (Refinement)

Given an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X, a *refinement* of $\mathcal{U}$ is an open cover $\mathcal{V} = \{V_j\}_{j\in J}$, such that there exists a function $\phi : I \to J$ for which

$$V_j \subset U_{\phi(i)}, \quad j \in J$$

holds. In words, each $V_j \in \mathcal{V}$ is contained in some $U_i \in \mathcal{U}$.

Definition 29.9: (Paracompact space)

A space X is called *paracompact* if any open cover of X admits a locally finite refinement.

Example 29.10: ($\mathbb{R}^n$ is Paracompact)

Suppose $\mathcal{U} = \{U_i\}_{i\in I}$ be an arbitrary open cover. Denote, $B_n = B_d(0, n)$ be the open ball of radius n, centered at orgin, and $\bar{B}_n$ be the closed ball. Note that each $\bar{B}_n$ is compact. Hence, for each n, there is a finte subset $I_n \subset I$ such that $\bar{B}_n \subset \bigcup_{i\in I_n} U_i$. Denote,

$$\mathcal{V}_1 := \{U_i \mid i \in I_1\}, \qquad \mathcal{V}_n := \left\{U_i \setminus \bar{B}_{n-1} \,\middle|\, i \in I_n\right\}, \quad n \geq 2.$$

Set, $\mathcal{V} = \bigcup \mathcal{V}_n$. By construction, each element of $\mathcal{V}$ is a subset of some $U_i \in \mathcal{U}$. For any $x \in \mathbb{R}^n$, consider $n \geq 1$ to be the least integer such that $x \in \bar{B}_n$. Then, $x \notin \bar{B}_{n-1}$. Clearly, we have $x \in U_i \setminus \bar{B}_{n-1}$ for some $i \in I_n$. Thus, $\mathcal{V}$ is a refinement of $\mathcal{U}$. More over, for any $x \in \mathbb{R}^n$, we have some $n \geq 1$ such that $x \in B_n$. It is clear that B_n can intersect only the open sets from $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n$, which is a finite collection. Thus, $\mathcal{V}$ is a locally finite refinement. Consequently, $\mathbb{R}^n$ is paracompact.

Exercise 29.11: (Exhaustion by Compacts)

A space X is said to be *exhaustible by compacts* if there are compact sets $K_n \subset X$ such that $X = \bigcup_{n\geq 1} K_n$, and $K_n \subset \mathring{K}_{n+1}$. Show that a T_2-space, which is exhaustible by compacts, is paracompact.

Remark 29.12: (Metric space is Paracompact)

Note that $\mathbb{R}$ with discrete topology is a metrizable space, which is not exhaustible by compacts, and hence, we cannot use the previous exercise! It is a deep theorem that any metric space is paracompact. The original proof was by Stone, which was simplified significantly by Mary Ellen Rudin.

Theorem 29.13: (M.E. Rudin's Proof : Metric Spaces are Paracompact)

A metrizable space is paracompact.

Proof

Let (X, d) be a metric space. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover. By the well-ordering principle, we assume that the indexing set Λ is well-ordered! Note that for any $x \in X$, there exists a least $\alpha \in \Lambda$ such that $x \in U_\alpha$, since Λ is a well-order and $\mathcal{U}$ is a cover.

By induction over n, we construct a locally finite refinement as follows. Firstly, for each $\alpha \in \Lambda$, define $A_{\alpha,n}$ to be the set of points $x \in X$, satisfying the following.

i) $\alpha \in \Lambda$ is the least index such that $x \in U_\alpha$.

ii) For any $j < n$, we have $d(x, y) \geq \frac{1}{2^j}$ whenever $y \in \bigcup_{\beta\in\Lambda} A_{\beta,j}$

iii) $B_d\left(x, \frac{3}{2^n}\right) \subset U_\alpha$.

Note that for $n = 1$, the second condition is vacuous, and thus $A_{\alpha,1}$ consists of $x \in X$ satisfying only the first and third condition. Moreover, at the n^{th}-step, the second condition does not involve any $A_{\alpha,n}$. Thus, one can inductively construct all $A_{\alpha,n}$. We allow the possibility that $A_{\alpha,n} = \emptyset$ for some $\alpha \in \Lambda$ and $n \geq 1$. Once these sets are constructed, whenever $A_{\alpha,n} \neq \emptyset$, denote

$$D_{\alpha,n} := \bigcup \left\{ B_d\left(x, \frac{1}{2^n}\right) \;\middle|\; x \in A_{\alpha,n} \right\}, \qquad \alpha \in \Lambda,\ n \geq 1.$$

If $A_{\alpha,n} = \emptyset$, set $D_{\alpha,n} = \emptyset$ as well. We claim that $\mathcal{D}$, the collection of all $D_{\alpha,n}$ as defined, is a locally finite refinement of $\mathcal{U}$.

Let us check $\mathcal{D}$ covers X. For any $x \in X$, there is a least $\alpha \in \Lambda$ such that $x \in U_\alpha$, and $x \notin U_\beta$ for all $\beta < \alpha$. Now, U_α is open, and hence, there is some $n \geq 1$ such that $B_d\left(x, \frac{3}{2^n}\right) \subset U_\alpha$. We claim that $x \in D_{\beta,j}$ for some $\beta \in \Lambda$ and some $j \leq n$. We have two possibilities. Suppose $x \in A_{\alpha,n}$. Then, clearly $x \in D_{\alpha,n}$ and we are done. Suppose $x \notin A_{\alpha,n}$. Since the first and third condition is satisfied, we must have that the second condition is violated. Thus, for some $j < n$, we have some $y \in A_{\beta,j}$ such that $d(x, y) < \frac{1}{2^j}$. But then, $x \in B_d\left(y, \frac{1}{2^j}\right) \subset D_{\beta,j}$. Thus, we see that $\mathcal{D}$ covers X.

By construction, each $D_{\alpha,n} \subset U_\alpha$, and hence, $\mathcal{D}$ is indeed a refinement of $\mathcal{U}$.

Finally, let us show that $\mathcal{D}$ is locally finite. Let $x \in X$. Get the least $\alpha \in \Lambda$ such that $x \in D_{\alpha,n}$ for some $n \geq 1$. Then, choose some $j \geq 1$ such that $B_d\left(x, \frac{1}{2^j}\right) \subset D_{\alpha,n}$. Fix the ball $U := B_d\left(x, \frac{1}{2^{n+j}}\right)$. We show the following.

a) For any $i \geq n+j$, we have $U \cap D_{\beta,i} = \emptyset$ for all $b \in \Lambda$.

b) For any $i < n+j$, we have $U \cap D_{\beta,i} \neq \emptyset$ for at most a single $\beta \in \Lambda$.

Let $i \geq n+j$. In particular, $i > n$. Fix some $y \in A_{\beta,i}$. We then have $d(y,z) \geq \frac{1}{2^n}$ whenever $z \in A_{\alpha,n}$, and hence, $y \notin D_{\alpha,n}$. As $B_d\left(x, \frac{1}{2^j}\right) \subset D_{\alpha,n}$, we then get $d(x,y) \geq \frac{1}{2^j}$ as well. Now, $i \geq j+1$ and $n+j \geq j+1$. Hence, it follows from triangle inequality that

$$B_d\left(x, \frac{1}{2^{n+j}}\right) \cap B_d\left(y, \frac{1}{2^i}\right) = \emptyset.$$

Indeed, if $z \in B_d\left(x, \frac{1}{2^{n+j}}\right) \cap B_d\left(y, \frac{1}{2^i}\right)$, then we have

$$d(x,y) \leq d(x,z) + d(z,y) < \frac{1}{2^{n+j}} + \frac{1}{2^j} \leq \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}} = \frac{1}{2^j},$$

a contradiction. Thus, for any $y \in A_{\beta,i}$, we have $U \cap B_d\left(y, \frac{1}{2^i}\right) = \emptyset$. But then clearly, $U \cap D_{\beta,i} = \emptyset$ holds for any $i \geq n+j$ and any $\beta \in \Lambda$.

Now, let $i < n+j$. Suppose $\beta \neq \gamma \in \Lambda$, without loss of generality, assume $\beta < \gamma$. Fix some $p \in D_{\beta,i}$ and $q \in D_{\gamma,i}$. Then, there are $y \in A_{\beta,i}, z \in A_{\gamma,i}$ such that $d(y,p) < \frac{1}{2^i}$ and $d(z,q) < \frac{1}{2^i}$. By construction, $B_d\left(y, \frac{3}{2^i}\right) \subset U_\beta$, and also, $z \notin U_\beta$ (as γ is the least one so that $z \in U_\gamma$). So, we must have $d(y,z) \geq \frac{3}{2^i}$. But then,

$$\frac{3}{2^i} \leq d(y,z) \leq d(y,p) + d(p,q) + d(q,z) < \frac{1}{2^i} + d(p,q) + \frac{1}{2^i} \Rightarrow d(p,q) > \frac{1}{2^i} \geq \frac{1}{2^{n+j-1}}.$$

Now, if U intersects both $D_{\beta,i}$ and $D_{\gamma,i}$ (with $\beta < \gamma$), then we can choose $p \in U \cap D_{\beta,i}$ and $q \in D_{\gamma,i}$. As argued above, we have $d(p,q) > \frac{1}{2^{n+j-1}}$. But, $p, q \in U = B_d\left(x, \frac{1}{2^{n+j}}\right)$. We have,

$$d(p,q) \leq d(p,z) + d(z,q) < \frac{1}{2^{n+j}} + \frac{1}{2^{n+j}} = \frac{1}{2^{n+j-1}},$$

a contradiction. Thus, U can intersect at most one $D_{\beta,i}$ whenever $i < n+j$.

But then it is clear U can intersect at most finitely many elements of $\mathcal{D}$, proving that $\mathcal{D}$ is a locally finite collection.

Thus, starting with the open cover $\mathcal{D}$, we have obtained a locally finite refinement $\mathcal{D}$ of $\mathcal{U}$. Consequently, any metric space is a paracompact space. □