Class Test 9

18th November, 2025

Q1. A subset $A \subset X$ is called *meager* if A can be written as a countable union of nowhere dense sets of X; otherwise A is called *non-meager*. A space X is called *Baire* if countable intersection of open dense sets is dense. A (locally) compact T_2 space is a Baire space. Prove the following. $[4+1+2+3=10]$

a) X is non-meager if and only if countable intersection of open dense sets is non-empty.

Solution: Suppose X is non-meager. Let G_n be a countable collection of open dense sets of X. Then, $F_n = X \setminus G_n$ is a closed set, which is nowhere dense. Consequently,

$$\emptyset \neq X \setminus \bigcup F_n = \bigcap (X \setminus F_n) = \bigcap G_n.$$

Thus, countable intersection of open dense sets are nonempty.

Conversely, suppose any countable intersection of open dense sets is nonempty in X. If possible, let X be meager. Then, $X = \bigcup A_n$ for nowhere dense sets A_n. It follows that $F_n = \overline{A_n}$ is a closed nowhere dense sets, and clearly, $X = \bigcup F_n$. But then $G_n = X \setminus F_n$ is an open dense set. We have, $\bigcap G_n = \bigcap (X \setminus F_n) = X \setminus \bigcup F_n = \emptyset$, a contradiction. Hence, X is non-meager.

b) A Baire space is non-meager (in itself).

Solution: Since in a Baire space countable intersection of open dense sets is dense, and thus, in particular nonempty, we have a Baire space is non-meager.

c) A subset of meager space is again meager.

Solution: Let $B \subset X$ be a meager set. Then, $B = \bigcup B_n$ for some nowhere dense sets B_n. Let $A \subset B$ be a subset. Consider $A_n = A \cap B_n$. Clearly, $A = \bigcup A_n$. We have,

$$A_n \subset B_n \Rightarrow \overline{A_n} \subset \overline{B_n} \Rightarrow \mathrm{int}\overline{A_n} \subset \mathrm{int}\overline{B_n} = \emptyset \Rightarrow \mathrm{int}\overline{A_n} = \emptyset.$$

Thus, A_n is nowhere dense, proving that A is a meager set.

d) $X = [0,1] \cup (\mathbb{Q} \cap [2,3])$ (as a subspace of $\mathbb{R}$) is a non-meager space, but not Baire.

Solution: Since $[0,1] \subset X$ is a compact T_2 space, it is Baire, and hence, non-meager. But then X must be non-meager, as any subset of a meager space is meager. Consider the countable collection open sets

$$G_q = X \setminus \{q\}, \quad q \in \mathbb{Q} \cap [2,3].$$

Clearly, each G_q is dense as well. Now, $\bigcap G_q = [0,1]$, which is not dense in X. Hence, X is not a Baire space.