Topology Course Notes (KSM1C03)

Day 28 : 14th November, 2025

`game of Choquet -- strongly Choquet space -- Baire space`

28.1 A digression: Game of Choquet

Given a space X, let us assume that two players are playing a game.

Round 0**:** Player I goes first by choosing an open set $U_0 \subset X$ and a point $x_0 \in U_0$. Then, player II chooses another open set V_0 satisfying $x_0 \in V_0 \subset U_0$.

Round 1**:** Player I now chooses an open set $U_1 \subset V_0$, and a point $x_1 \in U_1$. Then, player II chooses another open set V_1 satisfying $x_1 \in V_1 \subset U_1$.

Round n**:** At this stage, player I chooses an open set $U_n \subset V_{n-1}$ and a point $x_n \in U_n$. Player II then chooses an open set V_n satisfying $x_n \in V_n \subset U_n$.

Thus, we have an infinite game that goes like this:

Player I :	(U_0, x_0)		(U_1, x_1)		$\dots$		(U_n, x_n)		$\dots$
Player II :		V_0		V_1		$\dots$		V_n	$\dots$

This game is known as the *strong game of Choquet*. The usual *game of Choquet* is played the same way, but player I does not choose any points $x_n \in U_n$ at any stage, and thus, player II does not care about the points either. Observe that

$$\bigcap_{n \geq 0} U_n = \bigcap_{n \geq 0} V_n.$$

We say *player II wins the game* if $\bigcap V_n \neq \emptyset$ at the end of the game. Conversely, player I wins the game if $\bigcap U_n = \emptyset$ at the end of the game.

> ***Remark 28.1: (Winning strategy)***
>
> To formalize the concept of winning strategy (for player II), let us consider the following. Given a space, $(X, \mathcal{T})$, let us consider the sets
>
> $$\mathcal{T}_* := \{U \in \mathcal{T} \mid U \neq \emptyset\}, \qquad \mathcal{S} := \{(U, x) \mid U \in \mathcal{T}_*,\ x \in U\}.$$
>
> Then, a *winning strategy for player II* is a map
>
> $$f : \mathcal{S} \to \mathcal{T}_*$$

such that the following holds.

i) For any $(U, x) \in \mathcal{S}$, we have
$$x \in f(U, x) \subset U.$$

ii) For any sequence $(U_n, x_n) \in \mathcal{S}$ defined inductively, such that,
$$U_0 \supset V_0 := f(U_0, x_0) \supset U_1 \supset V_1 := f(U_1, x_1) \supset \cdots \supset U_n \supset V_n := f(U_n, x_n) \supset \ldots,$$
we always have $\bigcap V_n \neq \emptyset$.

Definition 28.2: (Strong) Choquet space

A space X is called a *Choquet space* (resp. *strongly Choquet space*) if in a game of Choquet (resp. *strong game of Choquet*), player II always has a winning strategy.

Remark 28.3

Winning a strong game of Choquet is more difficult for player II, as at the n^{th}-stage they have to choose an open set $V_n \subset U_n$ satisfying the extra condition $x_n \in V_n$. Thus, a strongly Choquet space is always a Choquet space. Also, since player I's goal is to make the intersection empty, player I is also denoted as player E (Empty). In this convention, player II is denoted as player N (Nonempty).

Proposition 28.4: (Completely metrizable space is strongly Choquet)

Let X be a completely metrizable space. Then X is strongly Choquet.

Proof

Let us fix a complete metric d on X, inducing the underlying topology. At the n^{th}-stage, after player E has chosen $x_n \in U_n \subset V_{n-1}$, player N chooses $x_n \in V_n \subset \overline{V_n} \subset U_n$, such that $\text{Diam}\overline{V_n} < \frac{1}{2^n}$. This is always possible in the metric space (X, d). Now, observe that
$$\bigcap U_n = \bigcap V_n = \bigcap \overline{V_n}.$$
But $\{\overline{V_n}\}$ is a decreasing sequence of closed sets in a complete metric space with diameter going to zero. Hence, $\bigcap \overline{V_n} \neq \emptyset$ (Check!). Thus, player N always wins. Hence, X is a strongly Choquet space. □

Theorem 28.5: (Strongly Choquet implies Complete Metrizability)

Suppose X is a metrizable space. If X is strongly Choquet, then X is completely metrizable.

Proof

Fix an arbitrary metric d on $(X, \mathcal{T})$, and consider the completion $(X, d) \hookrightarrow (X^\star, d^\star)$. We shall show that X is G_δ in $X^\star$.

Let us fix a winning strategy player N, and denote it by

$$\mathsf{f} : \{(U,x) \mid x \in U \in \mathcal{T}\} \longrightarrow \{U \in \mathcal{T} \mid U \neq \emptyset\}.$$

For each $n \geq 1$, let us consider $\mathcal{W}_n$ to be the collection of open sets $W \subset X^\star$, such that for some $x \in U \subset X$ we have

i) $U = X \cap \tilde{U}$, for some $\tilde{U} \subset X^\star$ open, with $\tilde{U} \subset B_{d^\star}\left(x, \frac{1}{n}\right)$,

ii) $W \cap X = \mathsf{f}(U,x)$, and

iii) $W \subset B_{d^\star}\left(x, \frac{1}{n}\right)$.

Denote,

$$G_n = \bigcup \{W \mid W \in \mathcal{W}_n\}.$$

Clearly $G_n \subset X^\star$ is open (possibly empty). Let us check that $X \subset G_n$. For any $x \in X$, let player E choose $U_0 = B_{d^\star}\left(x, \frac{1}{n}\right) \cap X$ and $x_0 = x$. At the n^{th}-stage, say player E chooses an open set $U_n = X \cap \tilde{U}_n$, where $x_n = x \in U_n$, and $\tilde{U}_n \subset B_{d^\star}\left(x, \frac{1}{n}\right)$. Then, player N chooses $V_n = \mathsf{f}(U_n, x)$, such that $x \in V_n \subset U_n$. But then, $V_n = X \cap W'$ for some $W' \subset X^\star$ open. Consider $W = W' \cap \tilde{U}_n$. Clearly,

$$X \cap W = X \cap \left(W' \cap \tilde{U}_n\right) = (X \cap W') \cap \left(X \cap \tilde{U}_n\right) = V_n \cap U_n = V_n = \mathsf{f}(U_n, x).$$

Also, $x \in W \subset B_{d^\star}\left(x, \frac{1}{n}\right)$. Thus, $x \in G_n$. Note that this argument requires *strong* game of Choquet. Consequently, we have $X \subset \bigcap G_n$.

Let us show that $\bigcap G_n \subset X$. Let $x \in \bigcap U_n$. For $n_1 = 1$, as $x \in G_{n_1}$, we have some $\tilde{V}_1 \in \mathcal{W}_{n_1}$ such that $x \in \tilde{V}_1$. Then, for some $y_1 \in U_1 \subset X$, we have $V_1 \coloneqq \tilde{V}_1 \cap X = \mathsf{f}(U_1, y_1)$, and moreover, $\tilde{V}_1 \subset B_{d^\star}\left(y_1, \frac{1}{n_1}\right)$. As $x \in \tilde{V}_1$, we have $\epsilon_1 \coloneqq d^\star(x, X^\star \setminus \tilde{V}_1) > 0$. Choose some $n_2 > n_1$ such that $\frac{1}{n_2} < \frac{\epsilon_1}{2}$. As $x \in G_{n_2}$, we have some $\tilde{V}_2 \in \mathcal{W}_{n_2}$ such that $x \in \tilde{V}_2$. Then, for some $y_2 \in U_2 \subset X$, we have $V_2 \coloneqq \tilde{V}_2 \cap X = \mathsf{f}(U_2, y_2)$, and moreover, $\tilde{V}_2 \subset B_{d^\star}\left(y_2, \frac{1}{n_2}\right)$. Note that $U_2 \subset \tilde{V}_1$. Indeed, $U_2 = X \cap \tilde{U}_2$ for some $\tilde{U} \subset X^\star$ open with $\tilde{U}_2 \subset B_{d^\star}\left(y_2, \frac{1}{n_2}\right)$. Then, for any $z \in U_2$ we have $d^\star(y_2, z) < \frac{1}{n_2}$. Also, we have $x \in \tilde{V}_2 \subset B_{d^\star}\left(y_2, \frac{1}{n_2}\right)$. Thus,

$$d(x,z) \leq d(x, y_2) + d(y_2, z) < \frac{1}{n_2} + \frac{1}{n_2} < \epsilon_1 = d^\star\left(x, X^\star \setminus \tilde{V}_1\right) \Rightarrow z \in \tilde{V}_1.$$

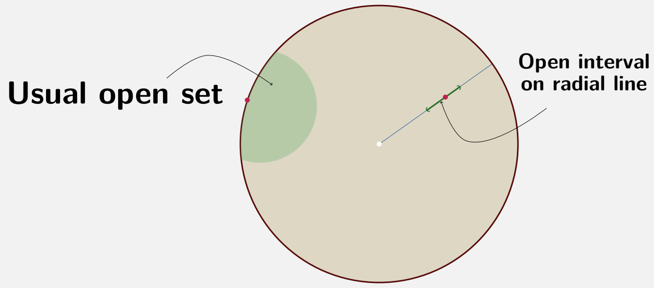
Thus, $U_2 \subset \tilde{V}_1$ holds, which implies $U_2 \subset V_1 = X \cap V_1$. Inductively, we continue this (strong) game of Choquet in a similar way. Since player N is playing by a winning strategy, it follows that $\bigcap U_n = \bigcap V_n \neq \emptyset$. Now, by construction, $x \in \bigcap \tilde{V}_n$. Since $(X^\star, d^\star)$ is a complete metric space, and since the diameters of $\tilde{V}_n$ are going to 0, it follows that $\bigcap \tilde{V}_n = \{x\}$, a singleton. But then,

$$\emptyset \neq \bigcap V_n = X \cap \bigcap \tilde{V}_n = X \cap \{x\} \Rightarrow x \in X.$$

Thus, we have $X = \bigcap G_n$, i.e, X is a G_δ-set in $X^\star$. Hence, X is completely metrizable. $\square$

Example 28.6: (Wheel with its Hub)

Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ be the closed unit disc with the center removed. Consider the collection of usual open sets in D as a subspace of $\mathbb{R}^2$, and additionally, consider every open intervals (in the usual sense) on every open radial line. It is easy to see, this collection is a basis for a topology on X. The space X is called the *wheel without its hub*.



Observe that X is not second countable, since the set $A = \left\{(x, y) \,\middle|\, x^2 + y^2 = \frac{1}{2}\right\}$ is a closed discrete subspace of X. Nevertheless, X is metrizable. Let us explicitly define a metric.

Consider the function $h : X \to [0, \infty)$ defined by $h(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} - 1$, and the function $r : X \to [0, \infty)$ defined by $r(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|}$. Here, for any $\mathbf{x} = (x, y)$, we have $\|\mathbf{x}\| = \sqrt{x^2 + y^2}$. It is easy to see that h, r are continuous maps. Define $d : X \times X \to [0, \infty)$ via

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} |h(\mathbf{x}) - h(\mathbf{y})|, & \text{if } r(\mathbf{x}) = r(\mathbf{y}), \\ h(\mathbf{x}) + h(\mathbf{y}) + \|r(\mathbf{x}) - r(\mathbf{y})\| & \text{otherwise.} \end{cases}$$

One can easily check that d is a metric on X, inducing the same topology (Check!). Moreover, one can show that d is a complete metric as well.

Let us instead play a strong game of Choquet on X! If at any stage, player E plays an open set U, and a point $\mathbf{x} \in U$ on some open radial line ℓ, then player N plays an open set V which is an open interval containing x on the radial line ℓ, such that the closed interval has length half that of the component of $\ell \cap U$ containing $\mathbf{x}$ (which is going to be interval), and is contained in said component. Then, we get a decreasing sequence closed intervals of ℓ with diameters going to 0. The intersection is nonempty by the completeness of $\mathbb{R}$, and so, player N wins. Suppose player E plays an open set U and a point $x \in U$ with x on the boundary circle. Then, player N plays a usual open neighborhood $V \subset U$ of x, such that $\bar{V} \subset U$. If player E never plays a point on any radial line (so the points are always on the circle), then we get a decreasing sequence closed sets in the boundary circle, which is a compact set. Thus, player N again wins. This proves X is strongly Choquet, and hence, completely metrizable.

Example 28.7: (Discrete rational extension of $\mathbb{R}$)

Consider X to be the discrete rational extension of $\mathbb{R}$, i.e, $X = \mathbb{R}$, with the topology $\mathcal{T}$ generated by the basis

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{Q},\ a < b\} \cup \{\{q\} \mid q \in \mathbb{Q}\}.$$

It is easy to see that $\mathcal{B}$ is a basis of clopen sets, and hence, X is a completely regular, second countable space, which is clearly T_1. By Urysohn's metrization theorem, X is then metrizable. Let us show that X is strongly Choquet.

If at any stage player E plays an open set U and a rational $q \in U$, player N can play $V = \{q\}$, and thereby winning the game. Suppose player E plays an open set $U \in \mathcal{T}$ and an irrational $x \in U$. Then, there is a finite length interval $x \in (a, b) \subset [a, b] \subset U$, such that $b - a < \frac{1}{2}\mathrm{Diam}U$. Player N chooses (a, b). Then, in a game, where player E never plays a rational point, we have $V_n = (a_n, b_n)$ for finite intervals, which are nested, with strictly decreasing diameter. In particular, $\bigcap V_n \neq \emptyset$, as $\mathbb{R}$ is complete. Thus, X is strongly Choquet. Consequently, the discrete rational extension of $\mathbb{R}$ is a completely metrizable space.

28.2 Baire Space

Definition 28.8: (Baire space)

A space X is called a *Baire space* if a countable intersection of dense, open sets of X is again dense.

Definition 28.9: (First and second category)

A subset $A \subset X$ is called of *first category* (or *meager*) if $A = \bigcup_{n \geq 1} A_n$ for some nowhere dense set $A_n \subset X$ (i.e., $\mathrm{int}\overline{A_n} = \emptyset$). If A cannot be written as the countable union of nowhere dense sets, then A is called of *second category* (or *non-meager*).

Exercise 28.10: (Subset of meager set)

Verify that a subset of a meager set is again meager.

Proposition 28.11

X is Baire if and only if countable union of closed nowhere dense sets have empty interior. In particular, a (nonempty) Baire space is non-meager (in itself).

Proof

Suppose X is a Baire space. Let A_n be a collection of closed nowhere dense sets. Then, $U_n = X \setminus A_n$ is a collection of open dense sets. We have $\bigcap U_n$ is dense. Now, for any nonempty open set $O \subset X$, we have $O \cap \bigcap U_n \neq \emptyset \Rightarrow O \not\subset X \setminus \bigcap U_n = \bigcup A_n$. Thus, $\bigcup A_n$ has empty interior.

Now, suppose countable union of closed nowhere dense sets in X has empty interior. Let U_n be a collection of open dense sets. Then, $A_n = X \setminus U_n$ is closed, nowhere dense. We have $\bigcup A_n$ has empty interior. So, for any nonempty open set $O \subset X$, we have $O \not\subset \bigcup A_n \Rightarrow O \cap (X \setminus \bigcup A_n) \neq \emptyset \Rightarrow O \cap \bigcap U_n \neq \emptyset$. Thus, $\bigcap U_n$ is dense. Hence, X is a Baire space.

Now, for a Baire space X, suppose $X = \bigcup A_n$ for some nowhere dense sets. Then, $X = \bigcup \overline{A_n}$, where $\overline{A_n}$ is closed, nowhere dense. But this contradicts that $\bigcup \overline{A_n}$ has empty interior. □

Remark 28.12: (Non-meager spaces need not be Baire)

There are non-meager spaces, which fail to be Baire. Consider $X = \mathbb{R} \times \{0\} \cup \mathbb{Q} \times \{1\} \subset \mathbb{R}^2$. Then, for each $q \in U$, we have $U_q \coloneqq X \setminus \{(q, 1)\}$, an open dense set. Clearly, $\bigcup U_n = \mathbb{R} \times \{0\}$ is not dense in X. Thus, X is not Baire. On the other hand, if possible, let us write $X = \bigcup A_n$ for some nowhere dense sets A_n. Then, $\mathbb{R} \times \{0\} = \bigcup (A_n \cap \mathbb{R} \times \{0\})$. Note that $B_n \coloneqq A_{\times}\mathbb{R} \times \{0\}$ is nowhere dense in $\mathbb{R} \times \{0\}$. But this implies $\mathbb{R} \cong \mathbb{R} \times \{0\}$ is a meager (in itself) space. This is a contradiction, as $\mathbb{R}$, being a completely metrizable space, is Baire, and hence, non-meager.