Class Test 8

11th November, 2025

Q1. On $\mathbb{R}$, consider the topology $\mathcal{T}$ (known as the *discrete rational extension of* $\mathbb{R}$) generated by the basis

$$\mathcal{B} := \{(a,b) \mid a,b \in \mathbb{R},\ a < b\} \cup \{\{q\} \mid q \in \mathbb{Q}\}.$$

Show that X is metrizable.

Solution: We shall use Urysohn's metrization theorem. Let us check the following.

X **is completely regular:** Consider the collection

$$\mathcal{I} = \{[p,q] \mid p,q \in \mathbb{Q},\ p \le q\}.$$

Clearly $X \setminus [p,q] = (-\infty,p) \cup (q,\infty)$ is open (in the usual topology), and thus, $[p,q]$ is closed for any $p \le q$. For $p = q$, we have $[p,q] = \{p\}$, which is open. For $p < q$, we have $[p,q] = (p,q) \cup \{p,q\}$, which is also open in $\mathcal{T}$. Thus, $\mathcal{I}$ is a collection of clopen sets.

For any open set $U \in \mathcal{T}$ and any $x \in U$, we have two possibilities. If $x \in \mathbb{Q}$, then we have $x \in \{x\} \subset U$, where $\{x\} \in \mathcal{I}$. On the other hand, if $x \notin \mathbb{Q}$, then there is some $(a,b) \in \mathcal{B}$ so that $x \in (a,b) \subset U$. But then there is some $r,s \in \mathbb{Q}$ so that $x \in [r,s] \subset (a,b) \subset U$. Clearly, $[r,s] \in \mathcal{I}$. Thus, $\mathcal{I}$ is a basis of $\mathcal{T}$

Since $\mathcal{I}$ is a basis of clopen sets, it follows that X is zero-dimensional. But then X is completely regular, and hence, regular.

X **is second countable:** The basis $\mathcal{I}$ constructed previously is countable. Hence, X is second countable.

X **is** T_3**:** It is clear that X is T_1, as any singleton is closed. Thus, X is T_3.

X **is metrizable:** Since X is second countable, T_3, it follows that X is metrizable.

Alternatively, you can also consider the countable collection

$$\mathcal{J} = \{(p,q) \mid p,q \in \mathbb{Q},\ p \le q\} \cup \{\{q\} \mid q \in \mathbb{Q}\},$$

and show that it is also a basis of clopen sets.

As an explicit metric, let us fix an enumeration $\mathbb{Q} = \{r_i\}_{i=1}^{\infty}$. Then, define $d : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by

$$d(x,y) = \begin{cases} \sup\left\{\frac{1}{i} \mid x \le r_i \le y\right\}, & x < y, \\ 0, & x = y, \\ \sup\left\{\frac{1}{i} \mid y \le r_i \le x\right\}, & y < x. \end{cases}$$

You may check that d is a metric on $\mathbb{R}$, inducing the topology $\mathcal{T}$. Indeed, for any $r \in \mathbb{Q}$, we have $r = r_i$ for some i, and then, $d(x,r_i) \ge \frac{1}{i}$ whenever $x \ne r_i$. Thus, $B_d(r,\epsilon) = \{r\}$ for any $\epsilon < \frac{1}{i}$. Also, for any $x < y < z$ it follows that $d(x,y) \le d(x,z)$, which implies that $B_d(x,\epsilon)$ is always an interval (possibly degenerate). This shows that the metric balls form a basis for the topology.