Topology Course Notes (KSM1C03)

Day 27 : 7th November, 2025

`product of complete metric space -- Lavrentieff's theorem -- completely metrizable and` G_δ

27.1 Product of metric spaces

Proposition 27.1: (Metric on Product Topology)

Suppose (X_i, d_i) is a countable collection of metric spaces. Let $X = \prod_{i=1}^{\infty} X_i$ be the product. Define

$$\rho_n(a,b) := \min\{d_n(a,b), 1\}, \ a, b \in X_n, \qquad \rho(x,y) := \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i}, \ x, y \in X.$$

Then, ρ is a metric on X, inducing the product topology.

Proof

Since each ρ_n is a bounded metric, it follows that ρ is well-defined. The metric properties can be checked easily. Let us show that the induced metric is the product topology. For some open $U \subset X_i$, consider the sub-basic open set $\mathcal{U} = \pi_i^{-1}(U)$. Without loss of generality, assume $U = B_{\rho_i}(x_i, r_i)$. Fix some $y \in U$. Set $\epsilon := \frac{r_i - \rho_i(x_i, y_i)}{2^i}$. Consider the metric ball $B_\rho(y, \epsilon)$. Then, for any $z \in B_\rho(y, \epsilon)$, we have

$$\begin{aligned} \rho_i(x_i, z_i) &\le \rho_i(x_i, y_i) + \rho_i(y_i, z_i) \\ &\le \rho_i(x_i, y_i) + 2^i \rho(y, z) \\ &< \rho_i(x_i, y_i) + (r_i - \rho_i(x_i, y_i)) = r_i \\ &\Rightarrow z_i \in U \Rightarrow z \in \mathcal{U}. \end{aligned}$$

Thus, $B_\rho(y, \epsilon) \subset \mathcal{U}$. This proves that the metric topology is finer than the product topology.

Conversely, consider a metric ball $B := B_\rho(x, \epsilon)$. Get some $N \ge 1$ with $\sum_{i>N} \frac{1}{2^i} < \frac{\epsilon}{2}$. Consider the set

$$V = \prod_{i=1}^{N} B_{\rho_i}\left(x_i, \frac{2^i \epsilon}{2N}\right) \times \prod_{i>N} X_i,$$

which is open in the product topology. Now, for any $y \in V$ we have

$$\rho(x,y) = \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i} \le \sum_{i=1}^{N} \frac{\frac{2^i \epsilon}{2N}}{2^i} + \sum_{i>N} \frac{1}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $V \subset B$. This proves that the product topology is finer than the metric topology. Hence, the two topologies coincide. □

Remark 27.2: (Arbitrary product of metric spaces)

Any uncountable product of (nonempty) metric space fails to be metrizable. In fact, the product topology fails to be first countable. There is a notion of *uniform metric* on an uncountable product, but the induced topology is strictly finer than the product topology, and strictly coarser than the box topology.

Theorem 27.3: (Countable product of completely metrizable spaces)

Let $\{X_n\}$ be a countable collection of nonempty spaces, and denote $X = \prod_{n=1}^{\infty} X_n$ be the product space. Then the following are equivalent.

a) X is completely metrizable.

b) X_n is completely metrizable for each $n \geq 1$.

Proof

Suppose X is completely metrizable. Fix some $a_i \in X_i$. Then, for each n, we have the subspace

$$X_n^{\star} = \{x \mid x_i = a_i \text{ if } i \neq n\} = \bigcap_{i \neq n} \pi_i^{-1}(a_i),$$

which is closed being the intersection of closed sets, and hence, completely metrizable. But X_n is homeomorphic to $X_n^{\star}$, and thus, X_n is completely metrizable as well.

Conversely, suppose each X_n is completely metrizable. Fix some complete metric d_n on X_n, and set

$$\rho_n(x, y) = \min\{d_n(x, y), 1\}, \quad x, y \in X_n.$$

Then, ρ_n is a bounded, complete metric, inducing the same topology. On $X = \prod X_n$, define

$$\rho(x, y) := \sum_{i=1}^{\infty} \frac{\rho_i(x_i, y_i)}{2^i}, \quad x, y \in X.$$

Then, ρ induces the product topology on X. Let us check that ρ is complete. Say, $\{x^n\} \subset X$ is a Cauchy sequence. Then, for a fixed i, consider the sequence $\{x_i^n\}_{n \geq 1} \subset X_n$. For $\epsilon > 0$, get $N \geq 1$ such that $\rho(x^n, x^m) < \frac{\epsilon}{2^i}$ for all $n, m \geq N$. Then, for $n, m \geq N$ we have

$$\rho_n(x_i^n, x_i^m) = 2^i \frac{\rho_n(x_i^n, x_i^m)}{2^i} \leq 2^i \rho(x^n, x^m) < \epsilon.$$

Thus, $\{x_i^n\} \subset X_i$ is a Cauchy sequence, and hence, converges to some $y_i \in X_i$. Consider the point $y = (y_i) \in X$. Fix some $\epsilon > 0$. Then, get some $K \geq 1$ such that $\sum_{n > N} \frac{1}{2^n} < \frac{\epsilon}{2}$. Also, for each $1 \leq i \leq K$, get some N_i such that

$$\rho_i(x_i^n, y_i) < \frac{2^i \cdot \epsilon}{2N}, \quad n \geq N_i.$$

Set $N = \max\{K, N_1, \dots, N_k\}$. Then, for $n \geq N$ we have

$$\rho(x^n, y) = \sum_{i=1}^{\infty} \frac{\rho_i(x_i^n, y_i)}{2^i} =\leq \sum_{i=1}^{N} \frac{\rho_i(x_i^n, y_i)}{2^i} + \sum_{i>N} \frac{1}{2^i} < N \cdot \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $x^n \to y$. Hence, (X, ρ) is a completely metric space. □

27.2 Lavrenthieff's Theorem

Proposition 27.4

Let X be a metrizable space, and Y be a completely metrizable space. Suppose, for some $A \subset X$, we have a continuous map $f : A \to Y$. Then, there exists a G_δ-set, say, $A^\star \subset X$ with $A \subset A^\star \subset \bar{A}$, and a continuous map $f^\star : A^\star \to Y$, which extends f.

Proof

Fix a complete metric d_Y on Y. For any $x \in \bar{A}$, denote the *oscillation*

$$\operatorname{osc}(f, x) := \inf\{\operatorname{Diam} f(U \cap A) \mid U \subset X \text{ is open, } x \in U\}.$$

As $x \in \bar{A}$, for any open neighborhood $x \in U$, we have $A \cap U \neq \emptyset$. Let us consider

$$A_n := \left\{x \in \bar{A} \;\middle|\; \operatorname{osc}(f, x) < \frac{1}{n}\right\}, \qquad A^* := \{x \in \bar{A} \mid \operatorname{osc}(f, x) = 0\}$$

Clearly $A^\star = \bigcap_{n \geq 1} A_n$. Moreover, for any $a \in A$, by continuity of f, we have some open $U \subset X$ such that $x \in U$ and $\operatorname{Diam} f(U \cap A) < \frac{1}{n}$. Thus, $a \in A_n$ for any $n \geq 1$. In particular, $A \subset A^\star \subset \bar{A}$ is clear.

Let us check that A_n is open in $\bar{A}$. For any $x \in A_n$, we have some open $U \subset X$ such that $x \in U$, and $\operatorname{Diam} f(U \cap A) < \frac{1}{n}$. But then for any $w \in U \cap \bar{A}$, it follows that $\operatorname{osc}(f, w) < \frac{1}{n}$. Thus, $x \in U \cap \bar{A} \subset A_n$. Since $x \in A_n$ is arbitrary, we have A_n is open in $\bar{A}$. Then, $A_n = \bar{A} \cap B_n$ for some open $B_n \subset X$. We have,

$$A^\star = \bigcap_{n \geq 1} A_n = \bigcap_{n \geq 1} \bar{A} \cap B_n = \bar{A} \cap \bigcap_{n \geq 1} B_n.$$

Since $\bar{A}$ is a closed set in a metric space, it is itself G_δ. Hence, we have $A^\star$ is a G_δ set in X.

Let us get a function $f^\star : A^\star \to Y$. For $x \in A^\star$, let $x_n \in A$ be a sequence with $\lim x_n = x$. Fix $\epsilon > 0$. Since $\operatorname{osc}(f, x) = 0$, we have some open set $U \subset X$ such that $x \in U$ and $\operatorname{Diam} f(U \cap A) < \epsilon$. As $x_n \to x$, we have some $N \geq 1$, such that for all $n, m \geq N$ we have $x_n, x_m \in U$. Then, it follows that $d_Y(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geq N$. In other words, $\{f(x_n)\}$ is a Cauchy sequence in (Y, d_Y). Since d_Y is complete, we have $f(x_n) \to y \in Y$. Set, $f^\star(x) = y$.

Let us check that $f^\star$ is well-defined. Suppose $z_n \in A$ is another sequence, with $z_n \to x \in A^\star$. Then, $\{f(z_n)\}$ is again Cauchy, and converges to some $w \in Y$. Fix some $\epsilon > 0$. Then, there is some $U \subset X$ open such that $x \in U$, and $\operatorname{Diam} f(U \cap A) < \frac{\epsilon}{3}$. As $\lim y_n = x = \lim z_n$, we have some $N \geq 1$, such that $y_n, z_n \in U$ for all $n \geq N$. Taking N larger, we may assume $d(f(y_n), y) < \frac{\epsilon}{3}$ and $d(f(z_n), w) < \frac{\epsilon}{3}$ for all $n \geq N$. Then, we have

$$d_Y(y, w) \leq d_Y(y, f(y_N)) + d_Y(f(y_N), f(z_N)) + d_Y(f(z_N), w) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since ϵ is arbitrary, it follows that $d_Y(y, w) = 0 \Rightarrow y = w$. Thus, $f^\star$ is well-defined.

Finally, let us check that $f^\star$ is a continuous extension. For any $a \in A$, we can consider the constant sequence $\{a_n = a\}$ that converges to a. Then, $f^\star(a) = \lim f(a_n) = \lim f(a) = f(a)$. Thus, $f^\star$ extends f. Let us check continuity. Let $x \in A^\star$, and fix $\epsilon > 0$. Then, there is some open set $U \subset X$ such that $\mathrm{Diam} f(U \cap A) < \frac{\epsilon}{3}$. Fix a sequence $y_n \in U \cap A$ such that $y_n \to y$. Now, for any $z \in U \cap A^\star$, consider a sequence $z_n \in U \cap A$ such that $z_n \to z$. There exists some $N \geq 1$ such that $d_Y(f(y_n), f^\star(y)) < \frac{\epsilon}{3}$ and $d_Y(f(z_n), f^\star(z)) < \frac{\epsilon}{3}$ for all $n \geq N$. We have,

$$d_Y(f^\star(y), f^\star(z)) \leq d_Y(f^\star(y), f(y_N)) + d_Y(f(y_N), f(z_N)) + d_Y(f(z_N), f^\star(z)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves $f^\star$ is continuous at y. Since $y \in A^\star$ is arbitrary, we have $f^\star : A^\star \to Y$ is a continuous extension. □

Theorem 27.5: (Lavrentieff's Theorem)

Suppose X, Y are completely metrizable spaces, and $f : A \to B$ is a homeomorphism, where $A \subset X, B \subset Y$. Then, f extends to a homeomorphism $f^\star : A^\star \to B^\star$, where $A^\star \subset X, B^\star \subset Y$ are G_δ-sets, with $A \subset A^\star \subset \bar{A}$ and $B \subset B^\star \subset \bar{B}$.

Proof

Let us denote $g = f^{-1}$. Since f, g are both continuous, we have G_δ-sets $A_1 \subset X, A_2 \subset Y$, with $A \subset A_1 \subset \bar{A}, B \subset B_1 \subset \bar{B}$, and extensions $f_1 : A_1 \to Y, g_1 : B_1 \to X$ of f and g respectively. Let us consider

$$A^\star := \{x \in A_1 \mid f_1(x) \in B_1\} = (f_1)^{-1}(B_1), \quad B^\star := \{x \in B_1 \mid g_1(x) \in A_1\} = (g_1)^{-1}(A_1).$$

Since these are inverse images of G_δ-sets, they are again G_δ. Clearly, $A \subset A^\star \subset \bar{A}$ and $B \subset B^\star \subset \bar{B}$. Let us denote $f^\star = f_1|_{A^\star}$ and $g^\star = g_1|_{B^\star}$. Clearly, $f^\star$ and $g^\star$ are continuous maps, extending f and g respectively. For any $x \in A^\star$, we have $f_1(x) \in B_1$, and so, $g_1 f_1(x) \in A_1$ is defined. Thus, $g_1 \circ f^\star : A^\star \to A_1$ is continuous. Say, $x_n \in A$ is a sequence, such that $x_n \to x \in A^\star$. Then,

$$g_1 f^\star(x) = \lim g_1 f^\star(x_n) = \lim g_1 f(x_n) = \lim g f(x_n) = \lim x_n = x.$$

Thus, $g_1 \circ f^\star : A^\star \to A^\star$ is the identity map. In particular, we have $g^\star \circ f^\star = \mathrm{Id}_{A^\star}$. Similarly, we have $f^\star \circ g^\star = \mathrm{Id}_{B^\star}$. Thus, $f^\star : A^\star \to B^\star$ is a homeomorphism, with inverse $g^\star : B^\star \to A^\star$. □

Theorem 27.6

Suppose X is a metrizable space, and $A \subset X$ is a completely metrizable space. Then, A is a G_δ-set in X.

Proof

Fix metric d on X. Consider $\iota : (X,d) \hookrightarrow (X^\star, d^\star)$ be the completion. Then, the restriction $f = \iota|_A : A \hookrightarrow X^\star$ is also an embedding, i.e, homeomorphism onto the image. Thus, we have a homeomorphism $A \supset A \to f(A) \subset X^\star$, where $A, X^\star$ are completely metrizable. By Lavrenteiff's theorem, f has an extension to a homeomorphism of G_δ sets of A and $X^\star$, containing A and $\iota(A)$ respectively. But then the extension must be ι itself, as on the left-hand side, the extended domain can only possibly be A. Thus, $f^\star(A^\star) = f(A) = \iota(A)$ is the extended set on the right-hand side. But then $\iota(A)$ is a G_δ set in $X^\star$. Taking inverse, it follows that A is then a G_δ set of X. □

Corollary 27.7: (Characterization of Completely Metrizable Space)

Given a metric space (X,d), the following are equivalent.

a) X is completely metrizable.

b) X is G_δ in the completion $X^\star$.

Corollary 27.8: ($\mathbb{Q}$ is not G_δ in $\mathbb{R}$)

$\mathbb{Q}$ is not G_δ in $\mathbb{R}$.