Topology Course Notes (KSM1C03)

Day 26 : 6th November, 2025

`completely metrizable space -- completion -- `G_δ`-subspace of completely metrizable space`

26.1 Completely metrizable space

Definition 26.1: (Cauchy sequence)

A sequence x_n in a metric space (X, d) is called a *Cauchy sequence* if given $\epsilon > 0$, there exists some $N = N_\epsilon \geq 1$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 26.2: (Complete metric space)

A metric space (X, d) is called *complete* if every Cauchy sequence in (X, d) converges.

Exercise 26.3

Given a metric space (X, d), we have a new metric $\bar{d}(x, y) = \min\{d(x, y), 1\}$, which is clearly bounded. Show that (X, d) is complete if and only if $(X, \bar{d})$ is complete.

Example 26.4

$\mathbb{R}$ with the usual metric is complete, but $X = (0, \infty)$ is not complete. Indeed, $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence (with the usual distance metric), which does not converge. On the other hand, consider

$$d(x, y) = |x - y| + \left|\frac{1}{x} - \frac{1}{y}\right|, \quad x, y \in X.$$

Check that d is a complete metric on X, inducing the same topology. Indeed, if $\{x_n\}$ is a Cauchy sequence in this metric, then both $\{x_n\}$ and $\left\{\frac{1}{x_n}\right\}$ are Cauchy in $\mathbb{R}$ with the usual metric, which implies $x_n \to c \neq 0$ (as we must have $\frac{1}{x_n} \to \frac{1}{c}$). Thus, $(0, \infty)$ is completely metrizable.

Example 26.5: ($\mathbb{Q}$ is not complete)

In $\mathbb{Q}$, consider the following sequence

$$x_1 = 1, \quad x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}, \; n \geq 1.$$

This sequence converges to $\sqrt{2}$ in $\mathbb{R}$, and hence, is a Cauchy sequence. Clearly, $\{x_n\} \subset \mathbb{Q}$ does not converge. Thus, $\mathbb{Q}$ is not complete with the usual metric.

Definition 26.6: (Completely metrizable space)

A space X is called a *completely metrizable space* if there exists a complete metric d on X inducing the topology.

Exercise 26.7

Check that complete metrizability is a topological property. That is, check that if X is homeomorphic to Y, and if Y is completely metrizable, then so is X.

Theorem 26.8: ($\mathbb{Q}$ is not completely metrizable)

A completely metrizable space, without any isolated point, is uncountable. Consequently, $\mathbb{Q}$ is not a completely metrizable space.

Proof

Suppose (X, d) is a complete metric space, without isolated points. Choose two distinct point $x_0, x_1 \in X$. This is possible, as X has no isolated point. Get open balls U_0, U_1 of radius ≤ 1 such that

$$x_0 \in U_0,\ x_1 \in U_1,\ \overline{U_0} \cap \overline{U_1} = \emptyset.$$

This is possible as X is T_3. Next, get more distinct points $x_{00}, x_{01} \in U_0 \setminus \{x_0\}$ and $x_{10}, x_{11} \in U_1 \setminus \{x_1\}$. Again, this is possible since there are no isolated points. Get open neighborhoods of radius $\leq \frac{1}{2}$

$$x_{00} \in U_{00} \subset \overline{U_{00}} \subset U_0,\ x_{01} \in U_{01} \subset \overline{U_{01}} \subset U_0, \quad x_{10} \in U_{10} \in \overline{U_{10}} \subset U_1,\ x_{11} \in U_{11} \subset \overline{U_{11}} \subset U_1,$$

with

$$\overline{U_{00}} \cap \overline{U_{01}} = \emptyset = \overline{U_{10}} \cap \overline{U_{11}}.$$

Inductively contitue getting points and open sets with disjoint closures. Thus, for any finite length word s formed by $\{0, 1\}$ we have a unique point x_s contained in an open set U_s of radius $\leq \frac{1}{|s|}$, where $|s|$ is the length of the word. Note that this is a countable infinite collection of points (and open sets), since the collection of all finite words formed by $\{0, 1\}$ is countable infinite. Moreover, for two distinct words s, t, if they are not sub-word of the other, then $\overline{U_s} \cap \overline{U_t} = \emptyset$. If $s \subset t$, then $\overline{U_t} \subset U_s$.

Let us now consider s to be an infinite word formed by $\{0, 1\}$. Denote s_n to be the initial word of s of length n, and set $x_n := x_{s_n}$. Let us check that $\{x_{s_n}\}$ is Cauchy. Let $\epsilon > 0$ be given, and fix $N \geq 1$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Observe that for any $n, m \geq N$, we have $x_n, x_m \in U_{s_N}$, and by construction, U_{s_N} is a ball with radius $\leq \frac{1}{|s_N|} = \frac{1}{N} < \frac{\epsilon}{2}$. Hence, $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Thus, $\{x_n\}$ is Cacuhy, which converges to a point, which we denote by x_s (where s is the infinite word).

Now, suppose s, t are two distinct infinite words of $\{0, 1\}$. Then, they differ at, say, the n^{th} position. But then $\overline{U_{s_{n+1}}} \cap \overline{U_{t_{n+1}}} = \emptyset$. This implies that $x_s \neq x_s$. Consequently, for each infinite word, we have unique point in X. Since the number of infinite words are uncountable (in fact, the cardinality is same as $\mathbb{R}$), it follows that X must be uncountable.

Since $\mathbb{Q}$ is a (metrizable) space without any isolated point, it cannot be completely metrizable. $\square$

26.2 Completion of a metric space

Definition 26.9: (Isometry)

A function $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is said to be an *isometry* if

$$d_Y(f(x), f(y)) = d_X(x, y), \quad \forall x, y \in X.$$

Definition 26.10: (Completion of a metric space)

Given a metric space (X, d_X), a complete metric space (Y, d_Y) is said to be a *completion* of X, if there exists an isometry $\iota : X \hookrightarrow Y$ such that the image $\iota(X)$ is dense in Y.

Theorem 26.11: (Completion : Existence and uniqueness)

Every metric space admits a completion, which is unique up to an isometry.

Proof

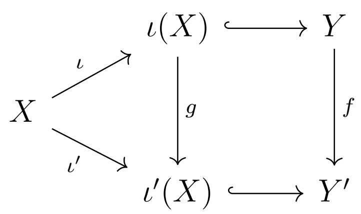
Let us first prove the uniqueness. Suppose, we have two completions $\iota : X \hookrightarrow Y$ and $\iota' : X \hookrightarrow Y'$. We have a well-defined continuous map

$$g := i' \circ \iota^{-1} : \iota(X) \to \iota'(X),$$

from a dense subset of Y to a dense subset of Y'. Note tha g is an isometry. Now, for any $y \in Y$, get a sequence $y_n \in \iota(X)$ such that $y_n \to y$. Then, $\{y_n\}$ is a Cacuchy sequence, and hence, so is $\{y'_n := g(y_n)\}$. Since Y' is complete, there is a point $y' \in Y'$ such that $y'_n \to y'$. Let us define $f(y) = y'$. We need to check that f is well-defined. Suppose $\{z_n\}$ is another sequence converging to y. Denote, $z'_n = g(z_n)$, and suppose $z'_n \to z' \in Y'$. Now,

$$d_{Y'}(y', z') = \lim d_{Y'}(y'_n, z'_n) = \lim d_{Y'}(g(y_n), g(z_n)) = \lim d_Y(y_n, z_n) = d_Y(y, y) = 0.$$

Thus, $y' = z'$, proving that f is well-defined.



Clearly f is surjective. Let us show that f is an isometry. Let $y, z \in Y$ be given. Suppose $y_n \to y, z_n \to z$, with $\{y_n\}, \{z_n\} \subset \iota(X)$. Denote, $y'_n = g(y_n), z'_n = g(z_n)$, and then, $y'_n \to y' = f(y), z'_n \to z' = f(z)$. We have,

$$d_{Y'}(f(y), f(z)) = d_{Y'}(y', z') = \lim d_{Y'}(y'_n, z'_n) = \lim d_Y(y_n, z_n) = d_Y(y, z).$$

Now, let us consider $h : Y' \to Y$ to be the isometry defined in the same way by using $\iota \circ (\iota')^{-1} : \iota'(X) \to \iota(X)$. Let us check that $h = f^{-1}$. It is clear that on points of $\iota(X)$, we have

$h \circ f = (i' \circ \iota^{-1}) \circ \left(\iota \circ (\iota')^{-1}\right) = \mathrm{Id}$. Now, for any $y \in Y$ we have $y = \lim y_n$ for some $y_n \in \iota(X)$. Then,

$$(h \circ f)(y) = h(f(\lim y_n)) = \lim h(f(y_n)) = \lim y_n = y.$$

Thus, $h \circ f = \mathrm{Id}_Y$. Similarly, $f \circ h = \mathrm{Id}_{Y'}$. Thus, we have $Y = Y'$ up to an isometry.

Let us now actually prove that a completion exists! The construction is similar to how one constructs $\mathbb{R}$ from $\mathbb{Q}$. Denote $\mathcal{C}(X)$ to be the collection of all Cauchy sequences in X. Note that given two Cauchy sequences $\{x_n\}, \{y_n\}$, we have $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and hence, converges. Indeed, for any $\epsilon > 0$, we have $N_1, N_2 \geq 1$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$ for $n, m \geq N$, and $d(y_n, y_m) < \frac{\epsilon}{2}$ for $n, m \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then, for any $n, m \geq N$ we have

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The first inequality follows from the triangle inequality and the symmetry! Now, define an equivalence relation $\sim$ on $\mathcal{C}(X)$ by

$$\{x_n\} \sim \{y_n\} \Leftrightarrow \lim d(x_n, y_n) = 0$$

Denote $X^\star = \mathcal{C}(X)/_\sim$ to be the collection of equivalence classes. Define $d^\star : X^\star \times X^\star \to X^\star$ by

$$d^\star([x_n], [y_n]) = \lim d(x_n, y_n).$$

Let us check that $d^\star$ is well-defined. Let $\{x'_n\}$ and $\{y'_n\}$ be some other representative. Then, we have

$$|d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n) \to 0, \quad \text{as } n- > \infty.$$

Hence, in the limit, we have $\lim d(x_n, y_n) = \lim d(x'_n, y'_n)$. It is easy to see that $d^\star$ is a metric on $X^\star$ (Check!). For any $x \in X$, define $\iota(x)$ to be the equivalence class of the constant sequence $\{x_n = x\}$. It follows that $\iota : X \hookrightarrow X^\star$ is an isometry (Check!).

Let us verify that $\iota(X)$ is dense in $X^\star$. Let $x^\star \in X^\star$ is represented by some Cauchy sequence $\{x_n\} \subset X$. Then, for any $\epsilon > 0$, there is some $N \geq 1$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$ for all $n, m \geq N$. Let $z = x_N$, and consider the point $\iota(z)$ formed by the constant sequence. Then,

$$d^\star(x^\star, \iota(z)) = \lim_n d(x_n, x_N) \leq \frac{\epsilon}{2} < \epsilon.$$

Since $\epsilon > 0$ and $x^\star$ is arbitrary, it follows that $\iota(X)$ is dense in $X^\star$.

Finally, we check that $d^\star$ is a complete metric. Let $\{z_n\}$ be a Cauchy sequence in $X^\star$. For $k \geq 1$, there is an $N_k \geq 1$ such that $d(z_n, z_m) < \frac{1}{k}$ for all $n, m \geq N$. For each N_k, we have some $w_k \in \iota(X)$ such that $d(w_k, z_{N_k}) < \frac{1}{k}$. Now, for any $\epsilon > 0$, choose some N such that $\frac{1}{N} < \frac{\epsilon}{3}$. Then, for $k, l \geq N$ we have

$$d^\star(w_k, w_l) \leq d^\star(w_k, z_{N_k}) + \underbrace{d^\star(z_{N_k}, z_{N_l})}_{<\max\{\frac{1}{k}, \frac{1}{l}\}} + d^\star(z_{N_l}, w_l) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

In other words, $\{w_k\}$ is a Cauchy sequence in $\iota(X)$. Without loss of generality, assume that each w_k represented as a constant sequence $w_k \in X$. Since ι is an isometry, it follows that $\{w_k\}$ is a Cauchy sequence in X, and hence, represents a point $w^\star \in X^\star$. We claim that the subsequence $\{z_{N_k}\}$ converges to $w^\star$. It is easy to see that $w_k \to w^\star$ (Check!). But then by construction, $z_{N_k} \to w^\star$. Since a subsequence of the Cacuchy sequence $\{z_n\}$ converges to $w^\star$, the Cauchy sequence $\{z_n\}$ also converges to $w^\star$. Thus, $X^\star$ is a completee metric space. In particular, completion of a metric space exists, unique up to isometry. □

Exercise 26.12

Fill in the details of the proof of the previous theorem.

Exercise 26.13

If X is a completely metrizable space, show that the completion $X^\star$ is homeomorphic to X.

26.3 Subspace of a completely metrizable space

Theorem 26.14: (G_δ-subspace of a completely metrizable space)

A G_δ subspace of a completely metrizable space is again completely metrizable.

Proof

Let (X, d) be a complete metric space. Fix an open set $U \subset X$. Then, we have a continuous function

$$\begin{aligned} f : U &\longrightarrow (0, \infty) \\ x &\longmapsto \frac{1}{d(x, X \setminus U)}. \end{aligned}$$

Since $X \setminus U$ is closed, the distance never vanishes, and thus f is indeed continuous. Let us now define $\rho : U \times U \to (0, \infty)$ by

$$\rho(x, y) = d(x, y) + |f(x) - f(y)|, \quad x, y \in U$$

It is easy to see that ρ is a metric on U. Moreover, ρ induces the subspace topology on U.

Let us show that (U, ρ) is complete. Say, $\{x_n\}$ is a Cauchy sequence in (U, ρ). Then, $\{x_n\}$ is Cauchy in (X, d) as well. Also, for any $\epsilon > 0$, there is some $N \geq 1$ such that for $n \geq N$ we have

$$|f(x_N) - f(x_n)| = \left| \frac{1}{d(x_N, X \setminus U)} - \frac{1}{d(x_n, X \setminus U)} \right| < \epsilon$$

Then, it follows that $d(x_n, X \setminus U)$ is bounded away from 0. In other words, there is some $\delta > 0$ such that

$$\{x_n\} \subset X_\delta := \{x \in X \mid d(x, X \setminus U) \geq \delta\} \subset U.$$

Now, X_δ is a closed subset, and hence, complete. Thus, we have $x_n \to x \in X_\delta \subset U$. Thus, (U, ρ) is a complete metric space.

Next, consider a G_δ-set $G = \bigcap_{n=1}^\infty U_n$, where $U_n \subset X$ is open. Now, U_n is completely metrizable, and hence, so is the product $\mathcal{U} = \prod_{n=1}^\infty U_n$. Insider $\mathcal{U}$ we have the diagonal,

$$\Delta_{\mathcal{U}} = \{x \in \mathcal{U} \mid x_i = x_j \ \forall i, j\}.$$

Note that $\Delta_{\mathcal{U}} = \Delta \cap \mathcal{U}$, where Δ is the diagonal in $\mathcal{X} = \prod_{n \geq 1} X$. Since Δ is closed in $\mathcal{X}$, it follows that Δ_U is closed in $\mathcal{U}$, and hence, completely metrizable. Now, the map

$$\begin{aligned} f : G &\longrightarrow \Delta_{\mathcal{U}} \\ x &\longmapsto (x, x, x, \dots) \end{aligned}$$

is clearly a continuous, bijection from G to Δ, with continuous inverse (given by any projection map). Indeed, it is the restriction of the usual diagonal map $X \hookrightarrow \mathcal{X}$. Thus, G is homeomorphic to Δ, and hence, G is completely metrizable. □

Example 26.15: (Irrationals are completely metrizable)

Since $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q \in \mathbb{Q}} (\mathbb{R} \setminus \{q\})$ is a G_δ-set in the complete metric space $\mathbb{R}$, it follows that the set of irrationals is a completely metrizable space.