Topology Course Notes (KSM1C03)

Day 25 : 5^{th} November, 2025

`Lebesgue number property -- Tietze extension`

25.1 Lebesgue number property

Definition 25.1: (Lebesgue number property)

A metric space (X, d) is said to have the *Lebsgue number property* if given any open cover $\{U_\alpha\}$, there is a number $\delta > 0$ (known as the *Lebesgue number of the covering*) such that for any set $A \subset X$ with $\mathrm{Diam} A < \delta$, we have $A \subset U_\alpha$ for some α.

Theorem 25.2: (Lebesgue number property and uniform continuity)

Let (X, d) be a metric space. Suppose every continuous map $f : X \to \mathbb{R}$ is uniformly continuous. Then, X has the Lebesgue number property

Proof

Suppose not. Then there exists an open cover $\mathcal{U} = \{U_\alpha\}$ without a Lebesgue number. Consequently, for each $n \geq 1$, there is a point x_n such that the set $A_n = B_d\left(x_n, \frac{1}{n}\right)$ is *not contained in any of the* U_α, i.e, $U_\alpha \setminus A_n \neq \emptyset$ for all α. Choose some $y_n \in A_n$ with $y_n \neq x_n$. Note that A_n is not singleton, otherwise $A_n \subset U_\alpha$ for some α, and so, y_n can always be chosen.

Let us observe that $\{x_n\}$ and $\{y_n\}$ has no convergent subsequence. If possible, suppose $x_{n_k} \to x$. Then, $x \in U_\alpha$ for some α. Now, there is some $\epsilon > 0$ such that $x \in B_d(x, 2\epsilon) \subset U_\alpha$. Since $x_{n_k} \to x$, there exists some $N \geq 1$ such that $x_{n_k} \in B_d(x, \epsilon) \subset B_d(x, 2\epsilon) \subset U_\alpha$ for all $n_k \geq N$. But then for some n_k sufficiently large, it follows from the triangle inequality that $A_{n_k} \subset B_d(X, 2\epsilon) \subset U_\alpha$, a contradiction. On the other hand, if $y_{n_k} \to y$, then it is clear that the subsequence $x_{n_k} \to y$, which is a contradiction. Thus, none of the sequences admit a convergent subsequence.

Next, we construct two disjoint closed sets from the two sequences. Set $x_{n_1} = x_1, y_{n_1} = y_1$. Clearly $\{x_{n_1}\} \cap \{x_{n_1}\} = \emptyset$. Choose $n_2 > n_1$, such that $x_{n_2} \neq x_{n_1}, y_{n_2} \neq y_{n_1}$, and $\{x_{n_1}, x_{n_2}\} \cap \{y_{n_1}, y_{n_2}\}$. This is possible, since otherwise the sequence will have to be eventually constant. Inductively, assume that we have constructed $\{x_{n_1}, \dots, x_{n_k}\}$ and $\{y_{n_1}, \dots, y_{n_k}\}$, which are disjoint sets of distinct points, with $n_1 < n_2 < \cdots < n_k$. Now, each of the points $\{x_{n_1}, \dots, x_{n_k}, y_{n_1}, \dots, y_{n_k}\}$

can only repeat finitely many times in $\{x_n\}$ and in $\{y_n\}$ (since otherwise there will be a convergent subsequence). Hence, we can choose $x_{n_{k+1}}, y_{n_{k+1}}$ at the induction step, so that $\{x_{n_1}, \dots, x_{n_{k+1}}\}, \{y_{n_1}, \dots, y_{n_{k+1}}\}$ are disjoint set of distinct points, with $n_{k+1} > n_k$. Set $A := \{x_{n_i}\}$ and $B := \{y_{n_i}\}$. By construction, $A \cap B = \emptyset$. Also, A, B are closed, since there are no (sub)sequential limits, and thus, A, B contains all of their limit points (which are none).

Now, (X, d) is a T_4-space. Hence, there is a continuous function $f : X \to \mathbb{R}$ with $f(A) = 0$ and $f(B) = 1$. We claim that f is not uniformly continuous. Indeed, for $\epsilon = \frac{1}{2}$ fixed, consider any $\delta > 0$ small. We must have some n_k with $\frac{1}{n_k} < \delta$. Now, $d(x_{n_k}, y_{n_k}) < \delta$, but $|f(x_{n_k}) - f(y_{n_k})| = |0 - 1| = 1 > \epsilon$. This contradicts the hypothesis. Hence, X has the Lebesgue number property. □

Exercise 25.3

Show that a metric space X has the Lebesgue number property if and only for any metric space Y any continuous map $f : X \to Y$ is uniformly continuous.

25.2 Tietze extension theorem

Theorem 25.4: (Tietze Extension Theorem)

A space X is normal if and only if given any closed set $A \subset X$ and continuous map $f : A \to \mathbb{R}$, there is an extension $\tilde{f} : X \to \mathbb{R}$, i.e, there is a continuous map $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(a) = f(a)$ for all $a \in A$.

Proof

Suppose X is normal. Firstly, let us consider a map $f : A \to [-1, 1]$. Define

$$A_1 :=:= \left\{x \in A \,\middle|\, f(x) \geq \frac{1}{3}\right\} = f^{-1}\left[\frac{1}{3}, 1\right], \quad B_1 := \left\{x \in A \,\middle|\, f(x) \leq -\frac{1}{3}\right\} = f^{-1}\left[-1, -\frac{1}{3}\right].$$

Clearly A_1, B_1 are disjoint closed sets of A, and hence, closed in X. As X is normal, by the Urysohn's lemma, we have continuous function $f_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that

$$f_1(A_1) = \frac{1}{3}, \quad f_1(B_1) = -\frac{1}{3}.$$

Now, for any $x \in A$ we have 3 cases.

a) $x \in A_1 \Rightarrow f_1(x) = \frac{1}{3}, f(x) \in \left[\frac{1}{3}, 1\right] \Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}$.

b) $x \in B_1 \Rightarrow f_1(x) = -\frac{1}{3}, f(x) \in \left[-1, -\frac{1}{3}\right] \Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}$.

c) $x \in A \setminus A_1 \cup B_1 \Rightarrow f_1(x), f(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right] \Rightarrow |f(x) - f_1(x)| \leq \frac{2}{3}$.

In other words, we have a continuous map $g_1 := f - f_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$. We repeat the process for g_1 instead of f. That is, we define $A_2 := g_1^{-1}\left[-\frac{2}{3}, -\frac{2}{9}\right]$ and $B_2 := g_1^{-1}\left[\frac{2}{9}, \frac{2}{3}\right]$. We get a function $f_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$, such that $f_2(A_2) = \frac{2}{9}, f_2(B_2) = -\frac{2}{9}$. Clearly, $|g_1 - f_2| \leq \left(\frac{2}{3}\right)^2$ on points of A. Define, $g_2 := g_1 - f_2 = f - f_1 - f_2$, clearly, $g_2 : A \to \left[-\frac{2}{9}, \frac{2}{9}\right]$. Inductively, we define $f_n : X \to \left[-\frac{2}{3^n}, \frac{2}{3^n}\right]$, such that

$$\left|f - \sum_{i=1}^{n} f_i\right| \leq \left(\frac{2}{3}\right)^n, \qquad \text{on points of } A.$$

Let us define $F(x) = \sum_{i=1}^{\infty} f_i(x)$. Observe that for any fixed $x \in X$, the series sum converges, since the partial sums

$$\left| \sum_{i=1}^{n} f_i(x) \right| \leq \sum_{i=1}^{n} \frac{2}{3^i}$$

are dominated by the geometric series. Moreover, for $a \in A$ we have,

$$\left| f(a) - \sum_{i=1}^{n} f_i(a) \right| \leq \left(\frac{2}{3}\right)^n \to 0 \Rightarrow F(a) = f(a).$$

In other words, F extends f. Let us show that F is continuous.

Fix some $x \in X$ and $\epsilon > 0$. Then, pick $N \geq 1$ such that $\sum_{n>N} \left(\frac{2}{3}\right)^n < \frac{\epsilon}{4}$. For $i = 1, \ldots, N$, using the continuity of f_i, pick open neighborhoods $x \in U_i \subset X$ such that

$$y \in U_i \Rightarrow |f_i(y) - f_i(x)| < \frac{\epsilon}{2N}.$$

Set $U = \bigcap_{i=1}^{N} U_i$, which is an open neighborhood of x. Then, for any $y \in U$ we have

$$\begin{aligned}
|F(y) - F(x)| &= \left| \sum_{i=1}^{\infty} f_i(y) - f_i(x) \right| \\
&\leq \sum_{i=1}^{N} |f_i(y) - f_i(x)| + \sum_{i>N} |f_i(y) - f_i(x)| \\
&< N \cdot \frac{\epsilon}{2N} + 2 \sum_{i>N} \left(\frac{2}{3}\right)^i \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Consequently, F is continuous at $x \in X$. Since x was arbitrary, we have the continuous extension $F : X \to [-1, 1]$ of $f : A \to [-1, 1]$.

Now, let us consider the general case. If $f : A \to [a, b]$ was given, we can use any homeomorphism $[a, b] \to [-1, 1]$ and its inverse, to get an extension $X \to [a, b]$. In case $f : A \to \mathbb{R}$ is given, we can use a homeomorphism $\mathbb{R} \to (-1, 1)$ to assume that the map is $f : A \to (-1, 1)$. Then, we end up with an extension $F_0 : X \to [-1, 1]$. Consider the set $A_0 = \{x \in X \mid F_0(x) \in \{\pm 1\}\} = F_0^{-1}(\{\pm 1\})$, which is clearly a closed set, disjoint from A. Then, by Urysohn's lemma, we have continuous map $\phi : X \to [0, 1]$ such that $\phi(A_0) = 0$ and $\phi(A) = 1$. Consider the function $F = \phi F_0$. Then, F is continuous, and clearly, $F(a) = F_0(a) = f(a)$ for any $a \in A$. Observe that $F : X \to (-1, 1)$. This concludes one direction of the proof.

Conversely, assume that given any closed $A \subset X$, and any continuous function $f : A \to \mathbb{R}$, there is a continuous extension $\tilde{f} : X \to \mathbb{R}$. Let $A, B \subset X$ be closed sets with $A \cap B = \emptyset$. Then, on the closed set $C = A \cup B$ consider the function $f_0 : C \to [0, 1]$ given by $f_0(a) = 0$ for all $a \in A$, and $f_0(b) = 1$ for all $b \in B$. Clearly it is continuous. Then, we have an extension $f : X \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$. By modifying the range of f, we can get the function $X \to [0, 1]$ as well. Thus, X is a normal space. □

Exercise 25.5

Assuming Tietze extension theorem, prove the Urysohn's lemma!