Class Test 7

4th November, 2025

Q1. On $\mathbb{R}^2$, consider $\mathcal{B}$ to be the collection of open discs (in the usual topology), with their horizontal diameters (except the center) removed, which is a basis for a topology, say, $\mathcal{T}$ on $\mathbb{R}^2$. Explicitly, a basic open set with center (x_0, y_0) and radius r is of the form

$$\left\{(x, y) \;\middle|\; (x - x_0)^2 + (y - y_0)^2 < r^2\right\} \setminus \left\{(x, y_0) \mid 0 < |x - x_0| < r\right\}.$$

The space $X = (\mathbb{R}^2, \mathcal{T})$ is called the *deleted diameter topology*.

a) Show that X is strictly finer than the usual topology on $\mathbb{R}^2$.

   Solution: Consider an arbitrary open set $U \subset \mathbb{R}^2$ in the usual topology. Then, for each $x \in U$, we have an open disc centered at x contained in U. Clearly, the same disc is a neighborhood of x in the deleted diameter topology after removing the deleted diameter. Thus, $U \in \mathcal{T}$.

   Any basic open set from $\mathcal{B}$ is not open in the usual topology on $\mathbb{R}^2$. Thus, $\mathcal{T}$ is a strictly finer topology.

b) Show that X is functionally Hausdorff.

   Solution: Consider the identity map $I : (\mathbb{R}^2, \mathcal{T}) \to (\mathbb{R}^2, \mathcal{T}_{\text{usual}})$. As the topology $\mathcal{T}$ is finer, we have I is continuous. Now, $\mathbb{R}^2$ is functionally Hausdorff, being metrizable. Hence, X is also functionally Hausdorff.

c) Show that X is not T_3.

   Solution: It is easy to see that the closure of a basic open set is simply a closed disc. Thus, given a basic open disc D centered at a point x, it follows that there is no open set $U \in \mathcal{T}$ such that $x \in U \subset \bar{U} \subset D$ (as there are no such basic open set). Hence, X is not regular, and consequently, not T_3.

d) Show that $Y = \mathbb{R} \times \{0\} \subset X$ is a closed discrete subspace.

   Solution: As Y is closed in the usual topology, it is closed in the finer topology. For any point p on Y, a basic open disc with center at p intersects Y precisely at p. Thus, Y is a closed discrete subspace.

e) Show that X is not Lindelöf.

   Solution: Any closed subspace of a Lindelöf space is again Lindelöf. But Y is a closed subspace, which is discrete and uncountable. Thus, Y cannot be Lindelöf. Hence, X is not Lindelöf.