Topology Course Notes (KSM1C03)

Day 24 : 31st October, 2025

product of normal space

24.1 Separation axioms : More properties and counterexamples

Proposition 24.1: ($T_5 \not\Rightarrow T_6$: The uncountable ordinal space $\overline{S_\Omega} = [0, \Omega]$)

The uncountable ordinal space $[0, \Omega]$ is a T_5-space, which is not T_6.

Proof

Since $[0, \Omega]$ is a linearly ordered space, we have $[0, \Omega]$ is T_5. Let us show that it is not G_δ. Consider $\{\Omega\}$, which is closed. If possible, suppose $\{\Omega\} = \bigcap_{n \geq 1} O_n$ for some open neighborhoods $\Omega \in O_n \subset [0, \Omega]$. Then, there is some $\alpha_n \in [0, \Omega)$ such that $\Omega \in (\alpha_n, \Omega] \subset O_n$. Since any countable collection of $[0, \Omega)$ is bounded above, we have some $\beta \in [0, \Omega)$ such that $\beta > \alpha_n$ for all $n \geq 1$. But then, $\{\Omega\} \subsetneq (\beta, \Omega] \subset \bigcap_{n \geq 1} O_n$. Thus, $\{\Omega\}$ fails to be a G_δ-set. Hence, $[0, \Omega]$ is not T_6. □

Remark 24.2

It is fact that the first uncountable ordinal $S_\Omega = [0, \Omega)$ is also not a G_δ-space, and hence, is not a T_6-space. Clearly, S_Ω, being a linearly ordered space, is T_5. Moreover, any ordinal space which is also a G_δ-space, is necessarily countable. Thus, all uncountable ordinal spaces are T_5 but not T_6.

Proposition 24.3: (Product of T_5 is not T_5)

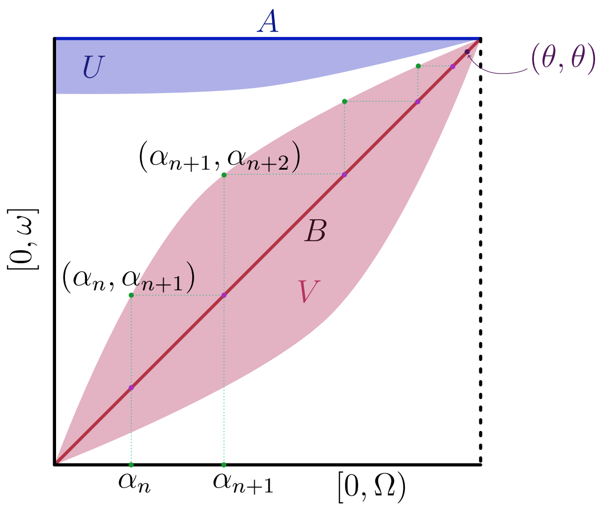
The product space $X = [0, \Omega) \times [0, \Omega]$ of two T_5 spaces is not T_5. In fact, the product is not even normal. Thus, product of T_4-spaces need not be T_4 either.

Proof

Since linearly ordered spaces are T_5, we have both $[0, \Omega)$ and $[0, \Omega]$ are T_5. Let us show that it fails to be normal. Consider

$$A := [0, \Omega) \times \{\Omega\}, \quad B := \{(\alpha, \alpha) \mid \alpha \in [0, \Omega)\}.$$

Note that A is the intersection of the closed set $[0, \Omega] \times \{\Omega\} \subset [0, \Omega] \times [0, \Omega]$ with the subspace $[0, \Omega) \times [0, \Omega]$. Similarly, B is the intersection of the diagonal $\Delta = \{(\alpha, \alpha) \mid \alpha \in [0, \Omega]\}$, which is closed in $[0, \Omega] \times [0, \Omega]$ as the space $[0, \Omega]$ is T_2, with the subspace X. Clearly, $A \cap B = \emptyset$. If possible, suppose there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.



For each $0 \leq \alpha < \Omega$, consider any $\alpha < \beta < \Omega$. If for all such β, we have $(\alpha, \beta) \in V$, then the limit (α, Ω) will be a limit point of V. But this contradicts $(\alpha, \Omega) \in U$ and $U \cap V = \emptyset$. Thus, there is some $\alpha < \beta < \Omega$ such that $(\alpha, \beta) \notin \Omega$. Let $\beta(\alpha)$ be the least such element, which exists as $[0, \Omega)$ is well-ordered. Let us now construct a sequence $\{\alpha_n\} \subset [0, \Omega)$ as follows. Start with $\alpha_1 = 0$. Then, set $\alpha_{n+1} = \beta(\alpha_n)$ for all $n \geq 1$. By construction, $\alpha_1 < \alpha_2 < \dots$. Let $\theta \in [0, \Omega)$ be the least upper bound of the sequence, and we have $\theta = \lim_n \alpha_n$. Then, $\lim_n (\alpha_n, \beta(\alpha_n)) = \lim_n (\alpha_n, \alpha_{n+1}) = (\theta, \theta) \in B \subset V$. But by construction, $(\alpha_n, \beta(\alpha_n)) \notin V$ for all $n \geq 1$. This is a contradiction. Hence, A, B cannot be separated by open neighborhoods. Thus, X is not normal, and hence, not T_5. □

Proposition 24.4: (Image of $T_{3\frac{1}{2}}$ need not be $T_{3\frac{1}{2}}$)

Continuous image of a $T_{3\frac{1}{2}}$-space need not be $T_{3\frac{1}{2}}$.

Proof

Recall the deleted Tychonoff plank $X = [0, \Omega] \times [0, \omega] \setminus \{(\Omega, \omega)\}$. In X, we have seen two closed sets $A = [0, \Omega) \times \{\omega\}$ and $B = \{\Omega\} \times [0, \omega)$, which are disjoint, but cannot be separated by open sets. Consider the quotient map $q : X \to X/A$. In X/A, observe that $q(B)$ is a closed set, since $q^{-1}(q(B)) = B$ is closed. Also, the point $a_0 = q(A)$ is not in $q(B)$. If possible, suppose there are open sets $U, V \subset X/A$ such that $a_0 \in U, A \subset V$ and $U \cap V = \emptyset$. Then, $A \subset q^{-1}(U), B \subset q^{-1}(V)$ are open sets such that $q^{-1}(U) \cap q^{-1}(V) = q^{-1}(U \cap V) = \emptyset$. This is a contradiction. Hence, X/A is not even regular, and in particular, not completely regular. □

24.2 Urysohn's metrization theorem

Proposition 24.5

Let X be a copletely regular space, and $\mathcal{B}$ be a fixed basis of X. Assume $\mathcal{B}$ is infinite. Then, there exists a family $\mathcal{F}$ of continuous functions $X \to [0, 1]$, with $|\mathcal{F}| \leq |\mathcal{B}|$, such that given any closed $A \subset X$ and $x \in X \setminus A$, there is a function $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(A) = 1$.

Proof

Given any pair of sets $(U, V) \in \mathcal{B} \times \mathcal{B}$, call it *good* if there is a continuous map $f : X \to [0, 1]$ such that $f(U) = 0$ and $f(X \setminus V) = 1$. Denote by $\mathcal{G}$ the collection of good pairs. Clearly,

$|\mathcal{G}| \leq |\mathcal{B} \times \mathcal{B}| = |\mathcal{B}|$. For each good pair $(U, V) \in \mathcal{G}$, choose a function $f_{U,V}$, and denote the family $\mathcal{F} = \{f_{U,V} \mid (U, V) \in \mathcal{B}\}$. Again, $|\mathcal{F}| = |\mathcal{G}| \leq |\mathcal{B}|$. We claim that $\mathcal{F}$ separates any closed set and a disjoint point.

Let $A \subset X$ be a closed set, and $x \in X \setminus A$ be a point. Get a basic open set $V \in \mathcal{B}$ such that $x \in V \subset X \setminus A$. By complete regularity, there is a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus V) = 1$. Now, $x \in f^{-1}\left[0, \frac{1}{2}\right)$ is an open neighborhood, so there is a basic open set $U \in \mathcal{B}$ such that $x \in U \subset f^{-1}\left[0, \frac{1}{2}\right)$. Construct the function $g : X \to [0, 1]$ by

$$g(y) = \begin{cases} 0, & f(y) \leq \frac{1}{2}, \\ 2\left(f(y) - \frac{1}{2}\right), & f(y) \geq \frac{1}{2}. \end{cases}$$

By pasting lemma, g is continuous. Moreover, $g(U) = 0, g(X \setminus V) = 1$. Thus, $(U, V) \in \mathcal{G}$ is a good pair. But then we have a $f_{U,V} \in \mathcal{F}$. Clearly, $f_{U,V}$ separates x and A, since $x \in U$ and $V \subset X \setminus A \Rightarrow A \subset X \setminus V$. □

Corollary 24.6

Let X be a second countable, completely regular space. Then there is a countable collection $\mathcal{F}$ of functions such that any closed set $A \subset X$ and any point $x \in X \setminus A$ can be separated by some function $f \in \mathcal{F}$.

Theorem 24.7: (Tychonoff embedding theorem)

Let X be a Tychonoff space (i.e, $T_{3\frac{1}{2}}$), and $\mathcal{B}$ be a fixed basis. Then, X is homeomorphic to a subspace of the cube $\mathcal{C} = [0, 1]^{|\mathcal{B}|}$

Proof

Get a family $\mathcal{F}$ of functions, with $|\mathcal{F}| \leq |\mathcal{B}|$. We prove an embedding $X \hookrightarrow [0, 1]^{|\mathcal{F}|}$, which is sufficient. Indeed, we have a map $\mathfrak{F} : X \to [0, 1]^{|\mathcal{F}|}$ defined by

$$\pi_f\left(\mathfrak{F}(x)\right) = f(x), \quad f \in \mathcal{F}, \qquad x \in X.$$

By the properties of the product topology, $\mathfrak{F}$ is continuous. As the space X is T_1, it follows that $\mathcal{F}$ separates points, and consequently, $\mathfrak{F}$ is injective. We show that $\mathfrak{F}$ is open onto its image.

Let $O \subset X$ be open, and $y \in \mathfrak{F}(O)$. Pick $x \in \mathfrak{F}^{-1}(y) \cap O$. Since $\mathcal{F}$ separates points and closed sets, there is some $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(X \setminus O) = 1$. Consider $W := \pi_f^{-1}([0, 1))$, which is open in the cube. Moreover, $W \cap \mathfrak{F}(X) \subset \mathfrak{F}(O)$. Indeed, for any $z \in Z$, with $\mathfrak{F}(z) \in W$, we must have $f(z) \neq 1 \Rightarrow z \notin X \setminus O \Rightarrow z \in O$, and thus, $\mathfrak{F}(z) \in \mathfrak{F}(O)$. In particular, $f(x) = 0 \Rightarrow y = \mathfrak{F}(x) \in W \Rightarrow y \in W \cap \mathfrak{F}(X) \subset \mathfrak{F}(O)$. As y was arbitrary, we have $\mathfrak{F}(O)$ is open. But then $\mathfrak{F}$ is a homeomorphism onto its image. In particular, X can be identified as a subspace of $[0, 1]^{|\mathcal{F}|}$. If $|\mathcal{F}| < \mathcal{B}$, then one can canonically see $[0, 1]^{|\mathcal{F}|}$ as a subspace of $[0, 1]^{|\mathcal{B}|}$. This concludes the proof. □

Theorem 24.8: (Urysohn's metrization theorem)

Any T_3, second countable space is metrizable.

Proof

Since X is second countable, it is Lindelöf. A regular, Lindelöf space is normal. Thus, X is T_4, and hence, $T_{3\frac{1}{2}}$. But then by the Tychonoff embedding theorem, X can be identified as a subspace of $[0,1]^\omega$, where $\omega = |\mathbb{N}|$. Now, $[0,1]^\omega$ is a metric space (being the countable product of metric spaces). Hence, X is a metric space. $\square$