Topology Course Notes (KSM1C03)

Day 23 : 30th October, 2025

T_4-space -- completely normal space -- T_5-space -- perfectly normal space -- T_6-space

23.1 T_4-space

Definition 23.1: (T_4-space)

A space X is called a *T_4-space* if it is normal and T_1.

Remark 23.2: (Normal + T_0 is not T_4)

As normal spaces are regular, $T_4 \Rightarrow T_3$. The excluded point topology on the three point set is normal, but not even T_1 (and hence, not T_2, T_3, T_4 either).

Proposition 23.3: ($T_4 \Rightarrow T_{3\frac{1}{2}}$)

Any T_4 space X is also a $T_{3\frac{1}{2}}$.

Proof

Let $A \subset X$ be a closed set, and $x \in X \setminus A$. Since X is T_1, we have $\{x\}$ is closed as well. Since X is normal, by Urysohn's lemma, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = 1$. But this means that X is completley regular. As X is T_0, we have X is $T_{3\frac{1}{2}}$. □

Proposition 23.4: (Compact $+ T_2 \Rightarrow T_4$)

A compact T_2 space X is T_4.

Proof

Let $A, B \subset X$ be disjoint closed sets. Fix some $a \in A$. Then, for each $b \in B$, there are open sets $U_{a,b}, V_{a,b}$ such that $a \in U_{a,b}, b \in V_{a,b}$ and $U_{a,b} \cap V_{a,b}$. Since B is closed in a compact space, B is compact. Thus, the cover $B \subset \bigcup_{b \in B} V_{a,b}$ has finite subcover $B \subset V_a := \bigcup_{i=1}^k V_{a,b_i}$. Then, $U_a := \bigcap_{i=1}^k U_{a,b_i}$ is an open set, with $a \in U_a$. Clearly, $U_a \cap V_a = \emptyset$. Now, we have a cover $A \subset \bigcup_{a \in A} U_a$, which again admits a finite subcover $A \subset U := \bigcup_{i=1}^l U_{a_i}$. We have an open set $V := \bigcap_{i=1}^l V_{a_i}$. Clearly, $B \subset V$ and $U \cap V = \emptyset$. Thus, we have that X is normal. Since X is T_2, we get X is T_4. □

Proposition 23.5: (Metrizable $\Rightarrow T_4$)

Metrizable spaces are T_4.

Proof

Fix a metric space (X, d). Let $A, B \subset X$ be disjoint closed sets. For each $a \in A$, fix $r_a := \frac{1}{3}d(a, B) > 0$, and for each $b \in B$, fix $s_b := \frac{1}{3}d(b, A)$. Consider the open sets

$$U := \bigcup_{a\in A} B_d(a, r_a), \qquad V := \bigcup_{b\in B} B_d(b, s_b).$$

Clearly, $A \subset U$ and $B \subset V$. If possible, suppose $z \in U \cap V$. Then, for some $a \in A$ and $b \in B$, we have

$$d(a, z) < r_a, \qquad d(b, z) < s_b.$$

Without loss of generality, assume $s_b \leq r_a$. Then,

$$3r_a = d(a, B) \leq d(a, b) \leq d(a, z) + d(z, b) < r_a + s_b \leq r_a + r_a = 2r_a,$$

a contradiction. Thus, $U \cap V = \emptyset$. Hence, X is normal. As X is T_2, we have X is T_4. $\square$

Proposition 23.6: ($T_{3\frac{1}{2}} \not\Rightarrow T_4$: Deleted Tychonoff plank)

The deleted Tychonoff plank $X := [0, \Omega] \times [0, \omega] \setminus \{(\Omega, \omega)\}$ is a $T_{3\frac{1}{2}}$ space, which is not T_4.

Proof

Recall that the ordinal spaces $[0, \Omega]$ and $[0, \omega]$ are compact, T_2, and hence, so is their product $T = [0, \Omega] \times [0, \omega]$. Thus, the Tychonoff plane T is T_4 and in particular, $T_{3\frac{1}{2}}$. Since being completely regular is hereditary (Check!), the subspace $X \subset T$ is $T_{3\frac{1}{2}}$.

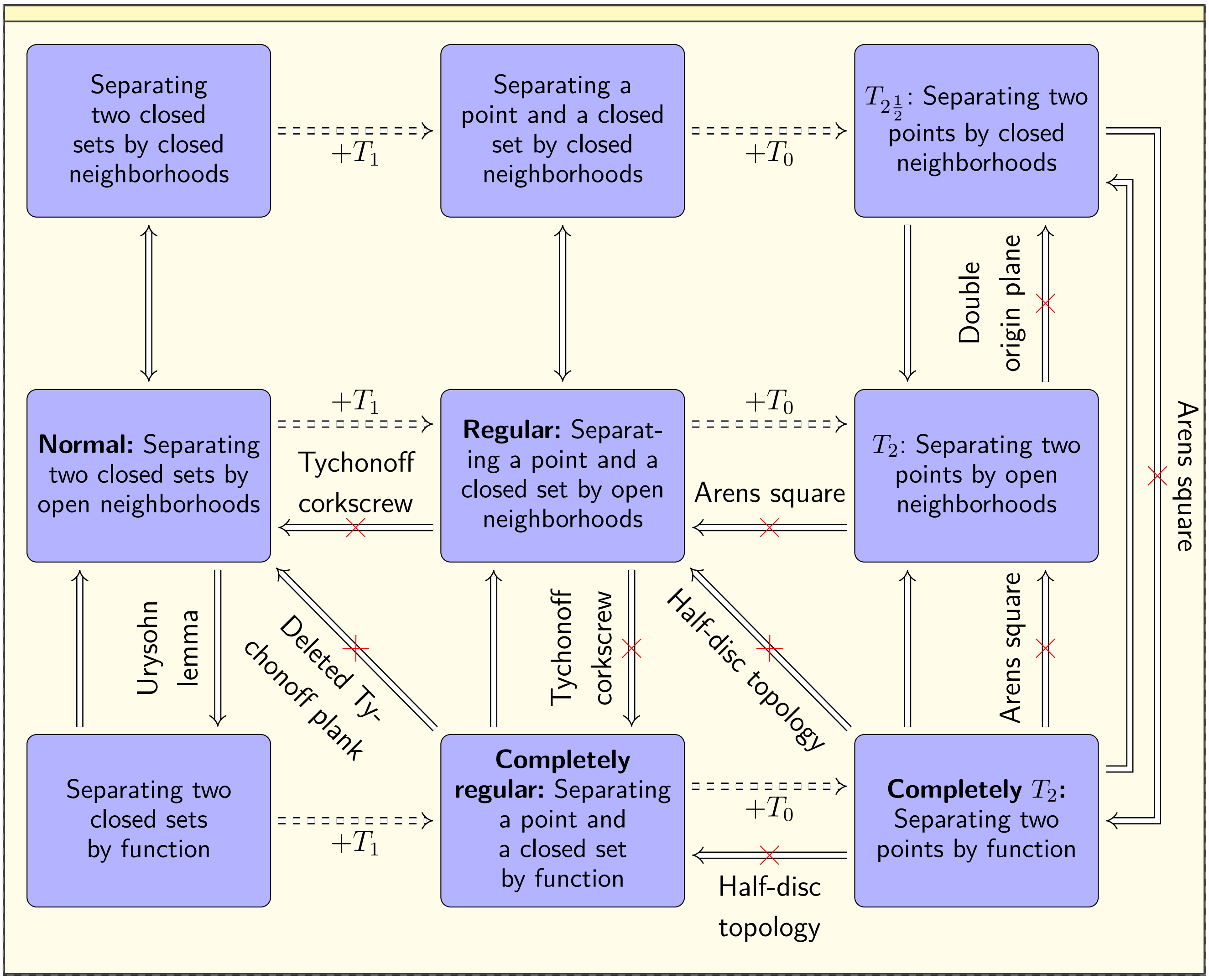
Let us show that X is not normal. Consider the sets $A = [0, \Omega) \times \{\omega\}$ and $B = \{\Omega\} \times [0, \omega)$, which are closed in the subspace topology of X. If possible, suppose there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. Then, for each $0 \leq n < \omega$, there is some $0 \leq \alpha_n < \Omega$ such that $(\alpha_n, \Omega] \times \{n\} \subset B$. Now $\{\alpha_n\}_n \subset [0, \Omega)$ is a countable set, and hence, there is an upper bound $\beta \in [0, \Omega)$. Then, we have the (open) set

$$(\beta, \Omega] \times [0, \omega) = \bigcup_{0\leq n<\omega} (\beta, \Omega] \times \{n\} \subset \bigcup_{0\leq n<\omega} (\alpha_n, \Omega] \times \{n\} \subset V.$$

Now, the basic open sets of $(\beta+1, \omega) \in A$ are of the form $(\gamma, \delta) \times (n, \omega)$, where $\beta + 1 \in (\gamma, \delta) \subset [0, \Omega)$ is an open interval. In particular, any open neighborhood of $(\beta + 1, \omega)$ will contain the set $\{\beta + 1\} \times [n, \omega)$ for some n large. Consequently, any open set containing $(b + 1, \omega)$ (and in particular, the open set U) will intersect the set V. This is a contradiction to $U \cap V = \emptyset$. Thus, X is not normal, and hence, not T_4. $\square$

Remark 23.7: (Separation axioms implications)

Let us summarize all the observations about separation axioms so far.



23.2 Completely normal and T_5-spaces

Definition 23.8: (Completely normal space)

A normal space is called a *completely normal space* (or *hereditarily normal space*) if every subspace is again a normal space.

Proposition 23.9

Given a space X, the following are equivalent.

a) X is completely normal.

b) Every open subset of X is normal.

c) Given any two subsets $A, B \subset X$, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

Proof

Suppose X is completely normal. Then, clearly any open set of X is again normal. Conversely, suppose every open set of X is normal. Let $Y \subset X$ be arbitrary subspace, and let $A, B \subset Y$ be closed sets with $A \cap B = \emptyset$. Note that $A = \overline{A}^Y = Y \cap \bar{A}$ and $B = \overline{B}^Y = Y \cap \bar{B}$. Consider the open set $W = X \setminus \bar{A} \cap \bar{B}$, which is normal. Now, $Y \cap (\bar{A} \cap \bar{B}) = (Y \cap \bar{A}) \cap (Y \cap \bar{B}) = A \cap B = \emptyset$. Thus, $Y \subset W$. Now, we have the closed sets $C = \bar{A} \cap W$ and $D = \bar{B} \cap W$ in the subspace W. Then, there are open sets $U, V \subset W$ (which are also open in X as W is open), such that $C \subset U, D \subset V$ and $U \cap V = \emptyset$. Then, we have

$$A = \bar{A} \cap Y \subset \bar{A} \cap W \subset U, B = \bar{B} \cap Y \subset \bar{B} \cap W \subset V.$$

Set $U' = U \cap Y, V' = V \cap Y$, which are open in Y, and clearly disjoint. Also, $A \subset U', B \subset V'$. Thus, Y is normal. Since Y was arbitrary, we have X is completely normal.

Next, let us assume X is completely normal. Let $A, B \subset X$ be arbitrary, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$. Consider $W = X \setminus \bar{A} \cap \bar{B}$. Then, W is normal. Also, $A \cap \bar{B} = \emptyset \Rightarrow A \subset X \setminus \bar{B} \subset W$, and similarly, $B \subset W$. Consider $C = W \cap \bar{A}$ and $D = W \cap \bar{B}$, which are closed in W. Note that $C \cap D = W \cap \bar{A} \cap \bar{B} = \emptyset$. Then, there are open sets $U, V \subset W$ (which are open in X, as W is open), such that $C \subset U, D \subset V$ and $U \cap V = \emptyset$. Clearly, $A \subset C \subset U, B \subset D \subset V$. Conversely, suppose given any two sets $A, B \subset X$ with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, we have open sets $U, V \subset X$ such that $A \subset U, B \subset V, U \cap V = \emptyset$. Let us show that X is completely normal. Fix some subspace $Y \subset X$, and closed sets $A, B \subset Y$ with $A \cap B = \emptyset$. Then, $A = Y \cap \bar{A}, B = Y \cap \bar{B}$. Now, $\bar{A} \cap B = \bar{A} \cap (B \cap Y) = (\bar{A} \cap Y) \cap B = A \cap B = \emptyset$, and similarly, $A \cap \bar{B} = \emptyset$. Then, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$. But then, consider $U' = Y \cap U, V' = Y \cap V$, which are open in Y. Clearly, $A \subset Y', B \subset V'$ and $U' \cap V' = \emptyset$. Thus, Y is normal. Since Y was arbitrary, we have X is completely normal. ☐

Definition 23.10: (T_5-space)

A completely normal, T_1 space is called a *T_5-space*.

Remark 23.11: ($T_4 \not\Rightarrow T_5$: Tychonoff plank)

Clearly $T_5 \Rightarrow T_4$. But the Tychonoff plank is a T_4-space, which is not T_5, since the (open) subspace deleted Tychonoff plank is not normal.

Theorem 23.12: (Order topology $\Rightarrow T_5$)

Any order topology is T_5.

Proof

Let $(X, \leq)$ be a totally ordered space, equipped with the order topology. Clearly X is T_2. Without loss of generality, assume that $|X| \geq 2$, so that even if X has end-points, they are distinct.

Let $A, B \subset X$ be arbitrary sets, with $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.

Step 1: Consider the set $Y = X \setminus (A \cup B)$. On Y, let us define an equivalence relation : $x \sim y$ if and only if the closed interval

$$[\min\{x, y\}, \max\{x, y\}] := \{z \in X \mid \min\{x, y\} \leq z \leq \max\{x, y\}\}$$

is contained in Y. Then, the equivalence classes represent the largest connected intervals in Y. By *axiom of choice*, let us choose a representative, say, $f(C)$ from each of the class C.

Step 2: For each $a \in A$, which is not the right end-point of X (if it exists at all), let us define $a < q_a$ as follows.

a) If a has an immediate successor in X, choose that to be q_a.

b) If a has no immediate successor, then for any $a < x$, we have $[a, x)$ contains a point of X. That is, a is then a *right* accumulation point. We consider two possibilities.

   i) Suppose a is a right accumulation point of A. Choose any $q_a \in A$ such that $a < q_a$ and $(a, q_a) \cap B = \emptyset$. This is possible since $A \cap \bar{B} = \emptyset$.

   ii) Suppose a is a right accumulation point of X, but not of A. In this case, consider $Z \coloneqq \{z \in A \cup B \mid z > x\}$. Since $A \cap \bar{B} = \emptyset$, we have some interval $[x, a) \cap Z = \emptyset$. Consequently, it follows that x is least upper bound of a unique component, say, C of Y. Let us take q_a to be the chosen point $f(C)$.

Observe that $[a, q_a)$ is always disjoint from B. Similarly, for each $a \in A$, which is not the left end-point of X, we choose $p_a < a$ as follows.

a) If a has an immediate predecessor in X, choose that to be p_a.

b) If a has no immediate predecessor in X, then a is a *left* accumulation point. We consider two possibilities.

   i) If a is an accumulation point of A, choose $p_a < a$ such that $(p_a, a) \cap B = \emptyset$.

   ii) If a is not an accumulation point of A, then as argued earlier, a is greatest lower bound of a unique component, say, C of Y. Take p_a to be the chosen point $f(C)$.

Note that a point $a \in A$ cannot be simultaneous both the end-points, since $|X| \geq 2$. Reversing the role of A and B, for each $b \in B$, we choose $p_b < b < q_b$ accordingly as well. Finally, for any $x \in A \cup B$, let us define the interval

$$I_x = (p_x, q_x) \quad \text{or,} \quad (p_x, x], \quad \text{or,} \quad [x, q_x),$$

as necessary. Clearly, for $a \in A$, we have I_a is an open neighborhood of a, disjoint from B. Similarly, for $b \in B$, we have I_b is an open neighborhood of b, disjoint from A.

Step 3: Say, $a \in A$ and $b \in B$ are fixed. Without loss of generality, assume $a < b$. Let us show that $I_a \cap I_b = \emptyset$. Suppose not. Then, $I_a \cap I_b = (p_b, q_a) \neq \emptyset$, and in particular, $p_b < q_a$. Clearly $b \notin I_a$, as $I_a \cap B = \emptyset$, and similarly, $a \notin I_b$. Thus, it follows that $a \leq p_b$ and $q_a \leq b$. Now, if q_a was the immediate successor of a, then, $I_a \cap I_b = (p_b, q_a) = \emptyset$. Thus, a must be defined by the other two cases (in particular, a is a right accumulation point). By the same argument, p_b is not the immediate predecessor of b, and consequently b is a left accumulation point. Now $p_b \notin B$, as otherwise $I_a \cap B \neq \emptyset$, and similarly, $q_a \notin A$. Thus, by previous step, p_b is not an accumulation point of B and q_a is not an accumulation point of A. Hence, there are components $C_1, C_2 \subset Y$ such that $(a, q_a) \subset C_1$ and $(p_b, b) \subset C_2$, where $q_a = f(C_1)$ and $p_b = f(C_2)$. Now,

$$\emptyset \neq I_a \cap I_b = (a, q_a) \cap (p_b, b) \subset C_1 \cap C_2.$$

Since C_1, C_2 are equivalence classes, the only possibility is $C_1 = C_2$, whence, $q_a = f(C_1) = f(C_2) = p_b$. But then, $I_a \cap I_b = \emptyset$, a contradiction.

Step 4: As a final step, consider the open sets

$$U := \bigcup_{a \in A} I_a, \qquad V := \bigcup_{b \in B} I_b.$$

Clearly, $A \subset U, B \subset V$. Moreover, $U \cap V = \emptyset$ by the previous step. Thus, X is perfectly normal. In particular, any linearly ordered space is T_5. □

Corollary 23.13: (Ordinal spaces are T_5)

Every ordinal space is T_5. In particular, $[0, \omega], [0, \Omega], [0, \Omega)$ are all T_5.

23.3 Perfectly normal and T_6-spaces

Definition 23.14: (Perfectly normal space)

A space X is called a *perfectly normal space* if given closed sets $A, B \subset X$ with $A \cap B = \emptyset$, there is a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. That is, a function *precisely* separates any two disjoint closed sets.

Theorem 23.15: (Vedenissoff's theorems)

Given a space X, the following are equivalent.

a) X is perfectly normal.

b) X is normal, and every closed set of C can be written as a countable intersection of closed sets (i.e, X is a G_δ-space).

c) Every closed set $A \subset X$ is the zero set of a continuous function, i.e, there is a continuous function $f : X \to \mathbb{R}$ such that $A = f^{-1}(0)$.

Proof

Suppose X is perfectly normal. Then clearly X is normal, as functional separation leads to separation by open neighborhoods. Let $C \subset X$ be an arbitrary closed set. We show that C is a G_δ-set, i.e, countable intersection of open sets of X. We have a continuous function $f : X \to [0,1]$ such that $f^{-1}(0) = C$ and $f^{-1}(1) = \emptyset$. Then, we have open sets $U_n := f^{-1}\left[0, \frac{1}{n}\right)$. Clearly, $C = \bigcap_{n \geq 1} U_n$. Thus, X is a normal, G_δ-space.

Next, suppose X is a normal, G_δ-space. Let $A \subset X$ be a closed set. Then, $A = \bigcap_{n \geq 1} U_n$ for some open sets $U_n \subset X$. Without loss of generality, we can assume that $U_{n+1} \subset U_n$ for each $n \geq 1$. Now, for each $n \geq 1$, we have disjoint closed sets A and $B_n := X \setminus U_n$. Then, as X is normal, by Urysohn's lemma we have a continuous map $f_n : X \to [0,1]$ such that $f_n(A) = \{0\}$

and $f_n(B_n) = \{1\}$. Consider a function $f : X \to [0,1]$ defined by

$$f(x) := \sum_{n\geq 1} \frac{f_n(x)}{2^{n+1}}, \qquad x \in X.$$

It follows that f is continuous. Clearly, $f(A) = 0$. Suppose $x \notin A$. Then, $x \notin U_{n_0}$ for some n_0. So, $x \in B_{n_0} \subset B_n$ for all $n \geq n_0$, and hence, $f_n(x) = 1$ for $n \geq 1$. We have

$$f(x) \geq \sum_{n\geq n_0} \frac{f_n(x)}{2^{n+1}} = \sum_{n\geq n_0} \frac{1}{2^{n+1}} = \frac{1}{2^{n_0+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) = \frac{1}{2^{n_0}} > 0.$$

Hence, $f^{-1}(0) = A$. As A was arbitrary closed set, this proves c).

Finally, suppose every closed set is the 0-set of some continuous function. Let $A, B \subset X$ be closed set with $A \cap B = \emptyset$. We have $f, g : X \to \mathbb{R}$ such that $f^{-1}(0) = A$ and $g^{-1}(0) = B$. As $A \cap B = \emptyset$, we have $f + g$ is nonvanishing. Consider the continuous function $h = \frac{f}{f+g}$. Clearly, $h : X \to [0,1]$. Also, $h(x) = 0 \Leftrightarrow f(x) = 0 \Leftrightarrow x \in A$, and $h(x) = 1 \Leftrightarrow f(x) = f(x) + g(x) \Leftrightarrow g(x) = 0 \Leftrightarrow x \in B$. Thus, $h^{-1}(0) = A$ and $h^{-1}(1) = B$. Hence, X is perfectly normal. □

Proposition 23.16: ($T_6 \Rightarrow T_5$)

Any subspace of a perfectly normal space is again perfectly normal. Consequently, a perfectly normal space is completely normal.

Proof

Let X be a perfectly normal space. Say, $Y \subset X$ is arbitrary subset, and $A \subset Y$ be closed. Then, $A = Y \cap \bar{A}$. We have a continuous function such that $\bar{A} = f^{-1}(0)$. Then, the restriction $g := f|_Y$ is again continuous, and clearly, $g^{-1}(0) = f^{-1}(0) \cap Y = \bar{A} \cap Y = A$. Thus, Y is perfectly normal, and hence, normal. In particular, X is completely normal. □

Definition 23.17: (T_6-space)

A space is called a *T_6-space* if it is perfectly normal, and T_1.

Proposition 23.18: (Metrizable $\Rightarrow T_6$)

Any metrizable space is T_6.

Proof

Fix a metric d on X. Given any closed sets $A, B \subset X$ with $A \cap B = \emptyset$, we have the continuous map

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)}, \qquad x \in X.$$

Then, $f^{-1}(0) = A$ and $f^{-1}(B)$. Clearly X is T_2 (and hence, T_1). Thus, X is T_6. □