Topology Course Notes (KSM1C03)

Day 22 : 29th October, 2025

`normal space -- Urysohn's lemma`

22.1 Normal space

Definition 22.1: (Normal space)

A space X is called a *normal space* if given any two disjoint closed sets $A, B \subset X$, there exists disjoing open sets separting them, i.e, there are open sets $U, V \subset X$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$

Remark 22.2

It is easy to see that normal spaces are regular. But completely regularity does not follow. Consider the space $X = \{-1, 0, 1\}$, with the topology $\mathcal{T} = \{\emptyset, X, \{-1\}, \{1\}, \{-1, 1\}\}$. This space is the *excluded point topology on the three point set*. It is easy to see that X is normal, since there are no disjoint nonempty closed sets! Indeed, the closed sets are $\{\emptyset, X, \{0\}, \{0, 1\}, \{0, -1\}\}$. Now, consider $A = \{0, 1\}$ and the point $x = -1 \in X \setminus A$. If possible, suppose $f : X \to [0, 1]$ is a continuous map, with $f(x) = 0$ and $f(A) = 1$. But then, $\{0\} = f^{-1}\left[0, \frac{1}{2}\right)$ must be open, a contradiction. Thus, X is not completely regular.

Proposition 22.3: (Normality by closed neighborhood)

X is normal if and only if given any closed set A and an open set $U \subset X$, with $A \subset U$, there exists an open set $V \subset X$ such that $A \subset V \subset \bar{V} \subset U$.

Proof

Suppose X is normal. Let $A \subset X$ be closed and $U \subset X$ be open, with $A \subset U$. Then, $B = X \setminus U$ is a closed set, disjoint from A. We have open sets $P, Q \subset X$ such that $A \subset P, B \subset Q$ and $P \cap Q = \emptyset$. Note that

$$P \subset X \setminus Q \Rightarrow \bar{P} \subset \overline{X \setminus Q} = X \setminus Q \subset X \setminus B = U.$$

That is, we have $A \subset P \subset \bar{P} \subset U$.

Conversely, suppose for any closed A and open U, with $A \subset U$, we have some open V such that $A \subset V \subset \bar{V} \subset U$. Let A, B be disjoint closed sets. Then, $A \subset X \setminus B$, which is open. Get open set U such that $A \subset U \subset \bar{U} \subset X \setminus B$. Let us take $V := X \setminus \overline{U}$, which is open. Then, $\bar{U} \subset X \setminus B \Rightarrow B \subset X \setminus \bar{U} = V$. Clearly, $U \cap V \subset \bar{U} \cap V = \emptyset \Rightarrow U \cap V = \emptyset$. Thus, X is a normal space. □

Exercise 22.4: (Normality is equivalent to separation by closed neighborhoods)

Check that a space X is normal if and only if for any closed sets $A, B \subset X$ with $A \cap B$, there are closed sets $P, Q \subset X$ such that $A \subset \mathring{P} \subset P, B \subset \mathring{Q} \subset Q$ and $P \cap Q = \emptyset$.

Theorem 22.5: (Urysohn's Lemma)

A space X is normal if and only if given disjoint closed sets $A, B \subset X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

Proof

Let X be a normal space. Fix two closed sets $A, B \subset X$ with $A \cap B = \emptyset$.

Step 1: Let us consider the dyadic rationals $D = \left\{ \frac{m}{2^n} \,\middle|\, m, n \geq 0,\ m \text{ odd} \right\} \cap (0, 1)$ in $[0, 1]$. For each $r \in D$, using the normality, we shall inductively construct an open set $U_r \subset X$ and a closed $V_r \subset X$, satisfying the following.

i) $A \subset U_r$ and $V_r \subset X \setminus B$ for all $r \in D$.

ii) $U_r \subset V_r$ for all $r \in D$.

iii) $V_r \subset U_s$ whenever $r < s$ in D.

Here are the first few steps of the induction.

$$
\begin{array}{ccccccccccccccc}
A & & & & & & & \subset & & & & & & & B^c \\
A & & & \subset & & & U_{\frac{1}{2}} & \subset & V_{\frac{1}{2}} & & & \subset & & & B^c \\
A & \subset & U_{\frac{1}{4}} & \subset & V_{\frac{1}{4}} & \subset & U_{\frac{1}{2}} & \subset & V_{\frac{1}{2}} & \subset & U_{\frac{3}{4}} & \subset & V_{\frac{3}{4}} & \subset & B^c
\end{array}
$$

Let us describe this formally. We induct over $n \geq 1$ where n appears as the exponent of 2 in $\frac{m}{2^n} \in D$, where $1 \leq m < 2^{k+1}$ are odd numbers. For notational convenience, let us denote $U_1 = B^c$ and $V_0 = A$.

Base case $n = 1$**:** We just have one value $\frac{1}{2}$ in this case. Since $A \subset B^c$, by normality, we have an open set $U_{\frac{1}{2}}$ and a closed set $V_{\frac{1}{2}} = \overline{U_{\frac{1}{2}}}$ such that $A \subset U_{\frac{1}{2}} \subset V_{\frac{1}{2}}$.

Inductive assumption $n = k$**:** Suppose, we for some $k \geq 1$, we have constructed the open and closed sets for all $\frac{m}{2^l} \in D$ with $l \leq k$.

Induction step $n = k + 1$**:** We need to get the sets labeled by $\left\{ \frac{1}{2^{k+1}}, \frac{3}{2^{k+1}}, \ldots, \frac{2^{k+1}-1}{2^{k+1}} \right\}$. But these appear in the middle of two sets already defined. As an example, for any $1 \leq m = 2l + 1 < 2^{k+1}$, we already have defined $V_{\frac{m-1}{2^{k+1}}} = V_{\frac{l}{2^k}} \subset U_{\frac{l+1}{2^k}} = U_{\frac{m+1}{2^{k+1}}}$ (after reducing the fractions $\frac{l}{2^k}$ and $\frac{l}{2^k}$ as needed, and noting, $V_0 = A, U_1 = B$ are the edge cases). Using normality, we get open and closed sets satisfying $V_{\frac{m-1}{2^{k+1}}} \subset U_{\frac{m}{2^{k+1}}} \subset V_{\frac{m}{2^{k+1}}} \subset U_{\frac{m}{2^{k+1}}}$.

Since every dyadic rational appears like this, we can construct the collection $\{U_r, V_r\}_{r\in D}$ with the desired properties.

Step 2: Let us now define a function $f : X \to [0,1]$ as follows.

a) Set $f(x) = 1$ if $x \notin U_r$ for all $r \in D$.

b) For any other x, define

$$f(x) = \inf\{r \in D \mid x \in U_r\}.$$

In particular, since $A \subset U_r$ for all r, we see that $f(x) = 0$ for $x \in A$. Similarly, as $U_r \subset V_r \subset X \setminus B \Rightarrow B \subset X \setminus U_r$ for all r, we see that $f(x) = 1$ for $x \in B$. Thus, f satisfies the desired properties. We need to show that f is continuous.

Step 3: Let us prove the continuity of the function defined in the previous step. We consider three cases.

a) Suppose $f(x) = 0$. If possible, suppose $x \notin U_{r_0}$ for some $r_0 \in D$. Then, for any $r \in D$ with $0 < r < r_0$, we must have $x \notin U_r$, as we have $U_r \subset V_r \subset U_{r_0}$. But this means $f(x) = \inf\{r \in D \mid x \in U_r\} \geq r_0 > 0$, a contradiction. Thus, $f(x) = 0 \Rightarrow x \in U_r$ for all $r \in D$. Now, for any open set $[0,\epsilon) \subset [0,1]$, we have some $r \in D \cap (0,\epsilon)$. Then, for any $y \in U_r$, we have $f(x) \leq r < \epsilon$. In other words, $x \in U_r \subset f^{-1}[0,\epsilon)$. Thus, f is continuous at x whenever $f(x) = 0$.

b) Suppose $f(x) = 1$. If possible, suppose $x \in V_{r_0}$ for some $r_0 \in D$. But then, $x \in U_r$ for any $r \in D$ with $r_0 < r$, and hence, $f(x) \leq r_0 < 1$, a contradiction. Thus, we have $f(x) = 1 \Rightarrow x \notin V_r$ for all $r \in D$. Now, for any open set $(1-\epsilon, 1] \subset [0,1]$, we have some $s \in D$ with $1 - \epsilon < s < 1$. Consider the open set $W = X \setminus V_s$. Clearly, $x \in W$. Then, for any $r < s$ in D, we have $U_r \subset V_r \subset U_s \subset V_s$. Thus, it follows that for any $y \notin V_s \Rightarrow y \not U_s$ we have $f(y) \geq r > 1 - \epsilon$. In other words, $x \in W \subset f^{-1}(1-\epsilon, 1]$. Thus, f is continuous at x whenever $f(x) = 1$.

c) Finally, suppose $0 < f(x) < 1$. Set $\delta := f(x)$, and get an open set $(\delta - \epsilon, \delta + \epsilon) \subset (0,1) \subset [0,1]$. Next, get $r_1, r_2 \in D$ satisfying $\delta - \epsilon < r_1 < \delta < r_2 < \delta + \epsilon$. Since D is dense in $(0,1)$, this is always possible. Consider the open set $W = U_{r_2} \setminus V_{r_1}$. Note that $f(x) = \delta < r_2 \Rightarrow x \in U_{r_2}$. Also, for any $r \in D$ with $r_1 < r < \delta$, we have $V_{r_1} \subset U_r$. Thus, $x \in V_{r_1} \Rightarrow y \in U_r \Rightarrow f(y) \leq r < \delta$, a contradiction. Thus, $x \in W$. Now, for any $r < r_1$, we have $U_r \subset V_{r_1}$, and thus, $y \in W \Rightarrow f(y) \geq r_1$. Also, $y \in W \subset U_{r_2} \Rightarrow f(y) \leq r_2$. Thus, for any $y \in W$ we have $f(y) \in [r_1, r_2] \subset (\delta - \epsilon, \delta + \epsilon)$. In other words, $x \in W \subset f^{-1}(\delta - \epsilon, \delta + \epsilon)$. Thus, f is continuous at x whenever $0 < f(x) < 1$.

Hence, we have proved that $f : X \to [0,1]$ is a continuous map. This concludes the theorem. □

Remark 22.6: (Onion Lemma!)

The construction in Uryshon's lemma has a resemblance of peeling an onion layer by layer: the space X is the onion, and any $U_r \setminus V_s$ for $s < r$ behaves like a layer. The function constructed in the lemma is called the *Urysohn's function* (for the sets A, B).