Class Test 6

28th October, 2025

Q1. Given a space X, define an equivalence relation : $x \sim y$ if and only if they have the same open neighborhoods. Denote the quotient space as $\mathcal{K}(X)$ (known as the *Kolmogorov quotient* of X).

a) Show that the quotient map $q : X \to \mathcal{K}(X)$ is both open and closed.

Solution: Let $U \subset X$ be open. Then, for any $x \in U$ and $y \notin U$, we have $x \not\sim y$. Consequently, $q^{-1}(q(U)) = U$. Hence, $q(U)$ is open in the quotient topology.

Similarly, let $C \subset X$ be closed. Again, for any $x \in C$ and $y \notin C$, we have $x \not\sim y$, as $X \setminus C$ is open. Thus, $q^{-1}(q(C)) = C$. Hence, $q(C)$ is closed in the quotient topology.

b) Show that $\mathcal{K}(X)$ is a T_0 space.

Solution: Let $[x], [y] \in \mathcal{K}(X)$ be two distinct points. Since $x \not\sim y$, without loss of generality, there is an open set $U \subset X$ such that $x \in U$ and $y \notin U$. But then $q(U)$ is an open set in $\mathcal{K}(X)$, with $[x] \in q(U)$. Also, $q^{-1}(q(U)) = U$, and hence $[y] \notin q(U)$. Thus, $\mathcal{K}(X)$ is T_0.

c) Show that X is regular if and only if $\mathcal{K}(X)$ is T_3.

Solution: Suppose X is regular. Let $A \subset \mathcal{K}(X)$ be a closed set and $[x] \in \mathcal{K}(X) \setminus A$ be a point. Now, we have the closed set $B = q^{-1}(A) \subset X$, and also, a point $x \in q^{-1}([x])$. Clearly, $x \notin B$, as otherwise $[x] \in A$. Hence, there are open sets $U, V \subset X$ such that $x \in U, B \subset V, U \cap V = \emptyset$. Since q is an open map, we have $q(U), q(V)$ are open in $\mathcal{K}(X)$, with $[x] \in q(U), q(B) = A \subset q(V)$, and $q(U) \cap q(V)$. Thus, $\mathcal{K}(X)$ is regular. Since $\mathcal{K}(X)$ is T_0, we see that $\mathcal{K}(X)$ is T_3.

Conversely, suppose $\mathcal{K}(X)$ is T_3. Let $A \subset X$ be closed and $x \in X \setminus A$. Then, $x \not\sim y$ for any $y \in A$. As q is a closed map, we have $q(A) \subset \mathcal{K}(X)$ is closed, and $[x] \notin q(A)$ is a point. Get open sets $U, V \subset \mathcal{K}(X)$ with $[x] \in U, q(A) \subset V, U \cap V = \emptyset$. Then, $x \in q^{-1}(U), A \subset q^{-1}(V)$ and $q^{-1}(U) \cap q^{-1}(V) = \emptyset$ gives a separation. Thus, X is regular.