Topology Course Notes (KSM1C03)

Day 21 : 24th October, 2025

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Tychonoff corkscrew property -- completely regular space
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21.1 Regular space and T_3 space (cont.)

Proposition 21.1: (Continuous map from S_Ω is eventually constant)

Given any continuous map $f : S_\Omega \to \mathbb{R}$, there exists some $\alpha \in S_\Omega$ such that $f(x) = c$ for all $x \geq \alpha$. Consequently, f can only have countably many distinct values.

Proof

If possible, suppose there exists some $\epsilon > 0$ such that for any $\alpha \in S_\Omega$ there exists some $\beta(\alpha) > \alpha$ with $|f(\alpha) - f(\beta)| \geq \epsilon$. Otherwise, for each $n \geq 1$, there exists some α_n such that for all $\beta > \alpha_n$, we have $|f(\beta) - f(\alpha_n)| < \frac{1}{n}$. If the sequence $\{\alpha_n\}$ is finite (i.e, there are finitely many points), then just take $\theta = \max \alpha_n$. It follows that for any $\beta > \theta$, we have $|f(\beta) - f(\theta)| < \frac{1}{n}$ for all n. In particular, $f(\beta) = f(\theta)$ for all $b > \theta$, proving the claim. If the sequence is not finite, without loss of generality, assume $\alpha_1 < \alpha_2 < \dots$. Now, recall that $[0, \Omega)$ is sequentially convergent. Hence, without loss of generality, the sequence $\{\alpha_n\}$ converges to some $\theta \in [0, \Omega)$, and $\theta \geq \alpha_i$ for all i. Then, by continuity of f we have $f(\theta) = \lim_n f(\alpha_n)$. Now, for any $\beta > \theta$, we have

$$|f(\beta) - f(\theta)| \leq |f(\beta) - f(\alpha_n)| + |f(\alpha_n) - f(\theta)| \to 0, \quad n \to \infty.$$

Thus, $f(\beta) = f(\theta)$ for any $\beta > \theta$, again proving the claim.

Thus, let us now assume that there exists some $\epsilon > 0$ such that for any $\alpha \in S_\Omega$ there exists some $\beta(\alpha) > \alpha$ with $|f(\alpha) - f(\beta)| \geq \epsilon$. Starting with $\alpha_0 = 0$, we can construct an increasing sequence $\alpha_0 < \alpha_1 < \dots$, where each α_j is inductively obtained as some $\beta(\alpha_{j-1})$. Now, $\{\alpha_j\}$ is a countable set, and hence, upper bounded. Suppose $\theta \in S_\Omega$ is the least upper bound of $\{\alpha_j\}$. Now, by continuity, we have some $\delta < \theta$ such that

$$f((\delta, \theta]) \subset \left(f(\theta) - \frac{\epsilon}{2}, f(\theta) + \frac{\epsilon}{2} \right).$$

Since θ is the least upper bound of the strictly increasing sequence α_j, there exists some $\delta < \alpha_{j_0} \leq \theta$. Now, for $\alpha_j < \alpha_{j+1} \leq \theta$. But then, $|f(\alpha_{j+1}) - f(\alpha_j)| < \epsilon$, a contradiction.
Hence, we have that there is some $\alpha \in S_\Omega$ such that $f(x)$ is constant for all $x \geq \alpha$. □

Proposition 21.2: ($T_3 \not\Rightarrow$ Completely T_2 : Tychonoff Corkscrew)

The Tychonoff corkscrew is T_3, but not completely T_2.

Proof

For any point other than $\alpha_\pm$, one can easily construct a basis of open sets which are regular (i.e, $\mathrm{int}(\bar{O}) = O$). Indeed, if the point is not on any of the "slits", we can take product of intervals. For a point on the slit, we might need to take the intervals in two different planks, but we can still get a basis of regular open sets. For α_+, the image of the basic open neighborhoods are open (Check!), and they are clearly regular open sets. Similar argument works for α_-. Thus, the Tychonoff corkscrew is a regular space. In fact, it is T_0 as every point is closed, and hence, T_3.

Let us now show that the space is not completely T_2. Suppose f is a real-valued continuous function. Observe that for $n \neq 0$, on each of the horizontal lines $A_\Omega \times \{n\} \times \{k\}$, the function f is constant on an interval of the form $[-\alpha, \alpha]$ about Ω. Same argument works for the x-axis as well, and we get a deleted neighborhood about $\{(\Omega, \omega, k)\}$ where f is constant. Now, there are countable infinitely many such intervals, on each of which f is constant. Indeed, on each stage, there are countable infinitely many horizontal lines (counting two lines for the x-axis), and there are countable infinitely many stages (the positive x-axes are getting counted twice, which is not an issue). Again, using the well-ordering, we can get a common α such that f is constant on each of the $[-\alpha, \alpha] \times \{\pm n\} \times \{k\}$ and on $([-\alpha, \alpha] \times \{\omega\} \setminus \{(\Omega, \omega)\}) \times \{k\}$, for all $k \in \mathbb{Z}$.

Fix some $-\beta \in [-\alpha, \Omega)$, and the corresponding $\beta \in (\Omega, \alpha]$. Then, denote the same points (i.e, their equivalence classes) in different stages as

$$-\beta^k = (-\beta, \omega, k), \quad \beta^k = (\beta, \omega, k).$$

Next, get the sequences

$$-\beta^k_{\pm n} = (-\beta, \pm n, k), \quad \beta^k_{\pm n} = (\beta, \pm n, k).$$

Clearly, as $\pm n \to \omega$, we have

$$-\beta^k_{\pm n} \to -\beta^k, \quad \beta^k_n \to \beta^k, \quad \beta^k_{-n} \to \beta^{k-1},$$

where the last convergence follows since the north edge of the fourth quadrant is identified with the south edge of the first quadrant of the stage just below! Now, $f\left(-\beta^k_{\pm n}\right) = f\left(\beta^k_{\pm n}\right)$. Hence, by continuity,

$$f(-\beta^k) = \lim f\left(-\beta^k_n\right) = \lim f\left(\beta^k_n\right) = f(\beta^k),$$

and also,

$$f(-\beta^k) = \lim f\left(-\beta^k_{-n}\right) = \lim f\left(\beta^k_{-n}\right) = f\left(\beta^{k-1}\right).$$

But then, inductively we see that $f(\pm\beta^k)$ are all constant. This implies that f is constant on the union of deleted intervals

$$\mathcal{I} = \bigcup_{k \in \mathbb{Z}} ([-\alpha, \alpha] \times \{\omega\} \setminus \{(\Omega, \omega)\}) \times \{k\}.$$

We can now get a sequence $\{a_i\}_{i=-\infty}^{\infty} \in \mathcal{I}$ (in fact, taking $a_{\pm i} = \pm\beta^i$ will do) such that $\lim_{i \to \infty} a_i = \alpha_+$ and $\lim_{i \to -\infty} a_i = \alpha_-$. This follows since the basic open neighborhoods of $\{\alpha_{pm}\}$ contains all the stages after (resp. below) a certain 'height'. By continuity of f, we have $f(\alpha_+) = f(\alpha_-)$. Thus, Tychonoff corkscrew is not functionally T_2, as no continuous function is able to distinguish the points $\alpha_\pm$. □

21.2 Completely regular space

Definition 21.3: (Completely regular space)

A space X is called a *completely regular space* if given any closed set $A \subset X$ and a point $x \in X \setminus A$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(A) = 1$.

Remark 21.4

It is immediate that a completely regular space is regular.

Definition 21.5: ($T_{3\frac{1}{2}}$-space)

A space X is called a *$T_{3\frac{1}{2}}$-space* (or a *Tychonoff space*) if it is completely regular, and T_0.

Remark 21.6

It is immediate that a $T_{3\frac{1}{2}}$-space is completely T_2, and hence, $T_{2\frac{1}{2}}$. Also, $T_{3\frac{1}{2}} \Rightarrow T_3$ is clear as well. Moreover, one can check that a completely regular space is $T_{3\frac{1}{2}}$ if and only if it is T_2. Thus, one can define $T_{3\frac{1}{2}}$-space as a completely regular, Hausdorff space.

Proposition 21.7: (Metrizable $\Rightarrow$ Tychonoff)

Metrizable spaces are Tychonoff.

Proof

Say (X, d) is a metric space. Let $A \subset X$ be closed, and $p \in X \setminus A$ be a point. Consider the map

$$f(x) := \frac{d(p,x)}{d(p,x) + d(A,x)}, \qquad x \in X.$$

It is easy to see that $f : X \to \mathbb{R}$ is continuous, and $f(p) = 0, f(A) = 1$. Thus, X is completely regular, and hence, Tychonoff. □

Proposition 21.8: ($T_3 \not\Rightarrow T_{3\frac{1}{2}}$: Tychonoff corkscrew)

The Tychonoff corkscrew X is T_3 but not $T_{3\frac{1}{2}}$.

Proof

We have seen that X is T_3 but not completely T_2. Since $T_{3\frac{1}{2}}$ implies completely T_2, it follows that X is not $T_{3\frac{1}{2}}$. □

Proposition 21.9: (Completely $T_2 \not\Rightarrow T_{3\frac{1}{2}}$: Half-disc topology)

The half-disc topology X is a completely T_2 space, which is not $T_{3\frac{1}{2}}$.

Proof

We have seen X is completely T_2 (as it was submetrizable), but not regular (in fact not even semiregular). Hence, X cannot be $T_{3\frac{1}{2}}$. □