Class Test 5

23rd October, 2025

Q1. Given any $A \subset X$, recall the boundary is defined as $\partial A := \bar{A} \cap \overline{X \setminus A}$, and the subset A is called nowhere dense if $\mathrm{int}(\bar{A}) = \emptyset$. Prove the following. $[2 \times 5 = 10]$

a) Suppose U is open in X. Show that ∂U is nowhere dense.

Solution : Note that boundary of any set is closed. Now, for any open set V, we have

$$V \subset \partial U = \bar{U} \cap \overline{X \setminus U} = \bar{U} \cap (X \setminus U) \Rightarrow V \cap U = \emptyset \Rightarrow V \cap \bar{U} = \emptyset \Rightarrow V = \emptyset.$$

Hence, $\mathrm{int}(\partial U) = \emptyset$, i.e., ∂U is nowhere dense.

b) Suppose C is closed in X. Show that ∂C is nowhere dense.

Solution : Suppose C is closed. Then, $U = X \setminus C$ is open. Now, $\partial C = \bar{C} \cap \overline{X \setminus C} = \overline{X \setminus U} \cap \bar{U} = \partial U$. Thus, ∂C is nowhere dense.

c) Give an example of some $A \subset X$, such that ∂A is not nowhere dense.

Solution : Consider $A = \mathbb{Q} \subset X = \mathbb{R}$, which is neither open nor closed. Now $\partial A = \mathbb{R}$, which is not nowhere dense.

d) Suppose $A \subset X$ is nowhere dense, and closed. Show that $A = \partial U$ for some $U \subset X$ open.

Solution : Since A is closed, consider $U = X \setminus A$, which is open. As A is nowhere dense, $U = X \setminus A$ is dense, i.e., $\bar{U} = X$. Then,

$$\partial U = \bar{U} \cap \overline{X \setminus U} = X \cap \bar{A} = X \cap A = A.$$

e) Give an example of some nowhere dense $A \subset X$ such that A is not a boundary of any open subset of X.

Solution : Consider $A = \left\{ \frac{1}{n} \;\middle|\; n \geq 1 \right\} \subset X = \mathbb{R}$. Then, $\mathrm{int}(\bar{A}) = \emptyset$, and thus, A is nowhere dense. On the other hand, A is not closed, and hence cannot be the boundary of any open set (or any subset) of X.