# Topology Course Notes (KSM1C03)

## Day 20 : 23rd October, 2025

`regular space -- ` $T_3$ ` space -- half-disc topology -- Tychonoff plank -- Tychonoff corkscrew`

### 20.1 Regular space and $T_3$-space

**Definition 20.1: (Regular space)**

A space $X$ is called *regular* if given any closed set $A \subset X$ and any point $x \in X \setminus A$, there exists open sets $U, V \subset X$ such that

$$x \in U, \quad A \subset V, \quad U \cap V = \emptyset.$$

**Proposition 20.2: (Regularity via closed neighborhood base)**

Given a space $X$, the following are equivalent.

a) $X$ is regular.

b) Given any $x \in X$ and open neighborhood $x \in U \subset X$, there exists a closed neighborhood $x \in \mathring{C} \subset C \subset U$.

c) Given any $x \in X$ and open neighborhood $x \in U \subset X$, there exists an open neighborhood $x \in V \subset \bar{V} \subset U$.

In other words, regularity is equivalent to the fact that closed neighborhoods of any point forms a local base at that point.

*Proof*

Suppose $X$ is regular. Let $x \in U \subset X$ be an open neighborhood. Then $A = X \setminus U$ is a closed set, and $x \notin A$. By regularity, there are open sets $P, Q \subset X$ such that

$$x \in P, \quad A \subset Q, \quad P \cap Q = \emptyset.$$

Note that

$$P \cap Q = \emptyset \Rightarrow P \subset X \setminus Q \Rightarrow \bar{P} \subset \overline{X \setminus Q} = X \setminus Q \subset X \setminus A = U.$$

Thus, we have a closed neighborhood $x \in P \subset \bar{P} \subset U$. This proves a) $\Rightarrow$ b).

Let us show b) $\Rightarrow$ c). Suppose $x \in U \subset X$ is given. Then, by b), we have some closed neighborhood $x \in \mathring{C} \subset C \subset U$. But then taking $V = \mathring{C}$, we have $x \in V \subset \bar{V} \subset \bar{C} = C \subset U$. This proves b) $\Rightarrow$ c).

Finally, suppose c) holds. Let $A \subset X$ be closed, and $x \notin A$ be a point. Then, $x \in U := X \setminus A$. By c), there is an open neighborhood such that $x \in V \subset \bar{V} \subset U$. Consider $P = V$ and $Q = X \setminus \bar{V}$. Then, $x \in V = P$, and $A = X \setminus U \subset X \setminus \bar{V} = Q$. Clearly, $P \cap Q = \emptyset$. Thus, $X$ is regular, proving a). □

**Definition 20.3: ($T_3$-space)**

A space $X$ is called a *$T_3$-space* if $X$ is regular and $T_0$.

**Example 20.4: (Regularity does not imply $T_3$)**

Consider $X = \{0, 1\}$ with the indiscrete topology. Then, $X$ is a regular space (in fact any indiscrete space is regular). But $X$ is not $T_0$. Thus, $X$ is not $T_3$.

**Proposition 20.5: ($T_3$ is equivalent to regular, $T_2$)**

A space $X$ is $T_3$ if and only if it is regular, $T_2$.

*Proof*

Suppose $X$ is regular, $T_2$. Since $T_2 \Rightarrow T_0$, we have $X$ is $T_3$. Conversely, suppose $X$ is $T_3$. Let us show that $X$ is $T_2$. Let $x \neq y \in X$. Since $X$ is $T_0$, there is an open set $U \subset X$, such that, without loss of generality, $x \in U$ and $y \notin U$. Then, there is an open neighborhood such that $x \in V \subset \bar{V} \subset U$. Take $W := X \setminus \bar{V}$. Then, $y \in X \setminus U \subset X \setminus \bar{V} = W$. Clearly, $V \cap W = \emptyset$. Thus, $X$ is $T_2$. □

**Proposition 20.6: ($T_3 \Rightarrow T_{2\frac{1}{2}}$)**

A $T_3$-space is $T_{2\frac{1}{2}}$.

*Proof*

Let $x \neq y \in X$. Since $X$ is $T_2$, we have open sets $U, V \subset X$ such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

But then there are open sets $A, B \subset X$ such that $x \in A \subset \bar{A} \subset U$ and $y \in B \subset \bar{B} \subset V$. Clearly, $\bar{A} \cap \bar{B} = \emptyset$. Thus, $X$ is $T_{2\frac{1}{2}}$. □

**Example 20.7: ($T_{2\frac{1}{2}} \not\Rightarrow T_3$ : Arens square is $T_{2\frac{1}{2}}$, but not regular)**

Recall that the Arens square $X$ is a $T_{2\frac{1}{2}}$-space. Let us show that $X$ is not regular. For the point $(0, 0)$, consider an open neighborhood $U_n$. But then for any basic open neighborhood $(0, 0) \in U_m \subset U_n$, we must have that $\overline{U_m}$ contains points with $y$-coordinate value $\frac{1}{4}$. Thus, $\overline{U_m} \not\subset U_n$. This means that the closed neighborhoods at $(0, 0)$ does not form a local base. Hence, $X$ is not regular.

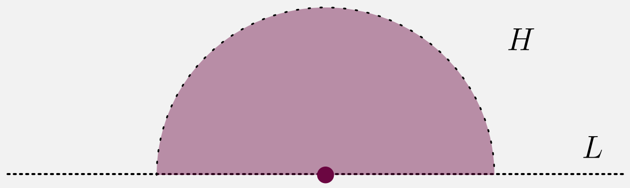
**Exercise 20.8**

Check that the double origin plane is not $T_3$.

**Example 20.9: (Half-disc topology)**

Consider the upper half plane $H = \{(x,y) \mid y > 0\}$ and the $x$-axis $L = \{(x,0) \mid x \in \mathbb{R}\}$. On the set $X := H \cup L$, consider the following topology.

- For any $(x,y) \in H$, consider the usual neighborhoods from $\mathbb{R}^2$ as the neighborhood basis.
- For $(x,0) \in L$, consider the open neighborhoods as $\{x\} \cup (H \cap U)$, where $U \subset \mathbb{R}^2$ is a usual open neighborhood of $(x,0)$.




This space $X$ is called the *half-disc topology*.

**Proposition 20.10: (Completely $T_2 \not\Rightarrow$ Regular : Half-disc topology)**

The half-disc topology $X$ is completely $T_2$, but not regular.

*Proof*

Observe that the inclusion map $\iota : X \hookrightarrow \mathbb{R}^2$ is continuous. Since $\mathbb{R}^2$ is a metric space, it is completely $T_2$. Consequently, it follows that $X$ is again completely $T_2$. Indeed, for any $x \neq y \in X$, we have $g : \mathbb{R}^2 \to [0,1]$ continuous such that $f(x) = 0$ and $f(y) = 1$. Then, $f := g \circ \iota : X \to [0,1]$ gives a functional separation.

Let us now show that $X$ is not regular (and hence not $T_3$ either). For any point $(x,0) \in L$, consider the half disc $D = H \cap B((x,0),\epsilon)$ of radius $\epsilon > 0$ and center $(x,0)$. Then, $U = \{(x,0)\} \cup D$ is an open set. These open sets clearly form a neighborhood basis at $(x,0)$. Observe that $\int \bar{U}$ contains all the points on the diameter of the half disc. Hence, we cannot find neighborhood basis of regular open sets at $(x,0)$ (recall : an open set $O$ is regular if $\operatorname{int}(\bar{O}) = O$). Thus, the half-disc topology is not regular. □

**Example 20.11: (Tychonoff Plank)**

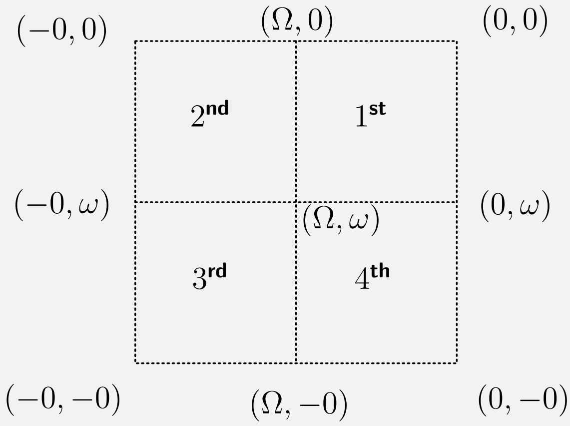
Recall the first infinite ordinal $\omega$ and the first uncountable ordinal $S_\Omega$. We get the well-ordered "intervals" $[0,\omega]$ (which you can think of as $\{0,1,2,\ldots,\omega\}$), and $[0,\Omega]$ (which you can think of as $\overline{S_\Omega} = S_\Omega \cup \{\Omega\}$). These are topological spaces equipped with the order topology, and in particular, they are compact. The *Tychonoff plank* is the product $[0,\Omega] \times [0,\omega]$. You can imagine this as the first quadrant of a coordinate grid : the $x$-axis corresponds to the first uncountable ordinal, whereas the $y$-axis corresponds to the first infinite ordinal. The *deleted Tychonoff plank* is the space $[0,\Omega] \times [0,\omega] \setminus \{(\Omega,\omega)\}$

**Example 20.12: (Corkscrew construction)**

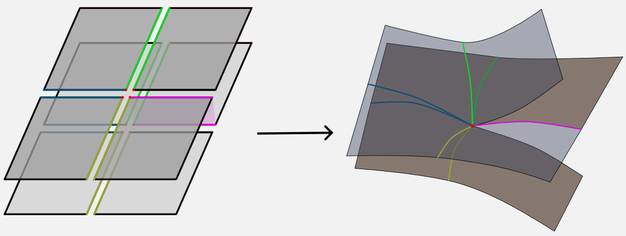
For the ordinal $\omega$ or $\Omega$, we have the totally ordered sets

$$A_\omega := [-0,-1,\ldots,\omega,\ldots,1,0], \quad A_\Omega := [-0,-1,\ldots,-\omega,\ldots,\Omega,\ldots,\omega,\ldots,1,0],$$

equipped with the order topology. Here, the negative of an element is a new element (so, $-0$ and $0$ different!). Taking product, we get a "coordinate plane", with all four quadrants a copy of Tychonoff plank.




Delete the "origin" $(\Omega, \omega)$. Now, take countable infinitely many copies of these planes (indexed by $\mathbb{Z}$), and stack them vertically. Next, cut all the planes along the positive $x$-axis. Then, along the cut, identify the north edge of the fourth quadrant of one plane to the south edge of the first quadrant of the *plane just below*. This is an identification space; since the origin was removed from all the planes, there is no issue about well-definedness.



This construction can be formalized as follows. For each $k \in \mathbb{Z}$, consider the following spaces

$$T_k^1 = ([\Omega, 0] \times [\omega, 0] \setminus \{(\Omega, \omega)\}) \times \{k\}, \qquad T_k^2 = ([-0, \Omega] \times [\omega, 0] \setminus \{(\Omega, \omega)\}) \times \{k\},$$
$$T_k^3 = ([-0, \Omega] \times [-0, \omega] \setminus \{(\Omega, \omega)\}) \times \{k\}, \qquad T_k^4 = ([\Omega, 0] \times [-0, \omega] \setminus \{(\Omega, \omega)\}) \times \{k\}.$$

These are copies of the deleted Tychonoff planks, representing the four quadrants at the $k^{\text{th}}$-stage. Let us identify the edges to make the corkscrew (see the picture above). We consider the set $X = \bigcup_{k \in \mathbb{Z}} (T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4)$, and on it define an equivalence relation as follows. For any $x \in X$, set $x \sim x$. Then, for each $k \in \mathbb{Z}$, consider the following collection of relations (and their reverse, to make it symmetric).

i) $x \sim y$ for $x = (\Omega, n, k) \in T_k^1$ and $y = (\Omega, n, k) \in T_k^2$ (identify the west-side of the first quadrant $T_k^1$ with the east-side of the second quadrant $T_k^2$, along the positive $y$-axis).

ii) $x \sim y$ for $x = (-\alpha, \omega, k) \in T_k^2$ and $y = (-\alpha, \omega, k) \in T_k^3$ (identify the south-side of the second quadrant $T_k^2$ with the north-side of the third quadrant $T_k^3$, along the negative $x$-axis).

iii) $x \sim y$ for $x = (\Omega, -n, k) \in T_k^3$ and $y = (\Omega, -n, k) \in T_k^4$ (identify the east-side of the third quadrant $T_k^3$ with the west-side of the fourth quadrant $T_k^4$, along the negative $y$-axis).

iv) $x \sim y$ for $x = (\alpha, \omega, k) \in T_k^4$ and $y = (\alpha, \omega, k-1) \in T_{k-1}^1$ (identify the north-side of the fourth quadrant $T_k^4$ with the south-side first quadrant $T_{k-1}^1$ **of the plane below**, along the positive $x$-axis).

The quotient space $X/_\sim$ looks like a corkscrew. This construction can be performed with other 'coordinate plane' whenever it makes sense!

**Example 20.13: (Tychonoff Corkscrew)**

Before performing the corkscrew construction as above with the Tychonoff planks, let us now add two extra points $\{\alpha_\pm\}$, and consider the space

$$Z = \{\alpha_+, \alpha_-\} \cup \bigcup_{k \in \mathbb{Z}} \left(T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4\right).$$

The topology on $Z$ is defined as follows. For any point $(\pm\alpha, \pm n, k)$, an open neighborhood basis is obtained from the induced topology of the deleted Tychonoff plank. Thus, basic open neighborhoods are products of intervals. For the point $\alpha_+$, a basic open neighborhood consist of all of $\bigcup_{k>i} (T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4)$ for some $i \in \mathbb{Z}$, i.e, everything above $i^{\text{th}}$-stage. Similarly, for $\alpha_-$, open neighborhoods consist of all of $\bigcup_{k<i} (T_k^1 \cup T_k^2 \cup T_k^3 \cup T_k^4)$ for some $i \in \mathbb{Z}$, i.e, everything below $i^{\text{th}}$-stage. It is easy to see that these collections of neighborhood bases forms a basis for a topology on $Z$. Let us now perform the identification as above, the points $\{\alpha_\pm\}$ are identified only to themselves,i.e, $\alpha_+ \sim \alpha_+$, $\alpha_- \sim \alpha_-$, and no other point. The quotient space $Z/_\sim$ is called the *Tychonoff corkscrew*.