Topology Course Notes (KSM1C03)

Day 19 : 21st October, 2025

$T_{2\frac{1}{2}}$-space -- completely T_2 space -- Arens square

19.1 $T_{2\frac{1}{2}}$-space and completely Hausdorff space

Definition 19.1: ($T_{2\frac{1}{2}}$-space)

A space X is called a *$T_{2\frac{1}{2}}$-space* (or a *Urysohn space*) if given any two distinct points $x, y \in X$, there exists disjoint closed neighborhoods of them, i.e, there are closed sets $A, B \subset X$ such that $x \in \mathring{A} \subset A, y \in \mathring{B}$ and $A \cap B = \emptyset$.

Remark 19.2: $T_{2\frac{1}{2}} \Rightarrow T_2$

Alternatively, we can define $T_{2\frac{1}{2}}$-space as follows : given any two distinct $x, y \in X$, there exists open sets $U, V \subset X$, such that $x \in U, y \in V$, and $\bar{U} \cap \bar{V} = \emptyset$. Thus, it is immediate that $T_{2\frac{1}{2}} \Rightarrow T_2$.

Example 19.3: ($T_2 \not\Rightarrow T_{2\frac{1}{2}}$)

Let us consider the *double origin plane*. Let X be $\mathbb{R}^2$, with an additional point 0^*. For any $x \in X$ with $x \neq 0, 0^*$, declare the open neighborhoods of x to be the usual open sets $x \in U \subset \mathbb{R}^2 \setminus \{0\}$. For the origin 0, declare the basic open neighborhoods

$$U_n := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < \frac{1}{n},\ y > 0 \right\} \cup \{0\}, \quad n \geq 1,$$

and similarly, for 0^*, declare the basic open neighborhoods to be

$$V_n := \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < \frac{1}{n},\ y < 0 \right\} \cup \{0^*\}, \quad n \geq 1.$$

It is easy to see that these basic open sets form a basis for a topology on X. With this topology, X is called the double origin plane. It is easy to see that X is T_2. But for any two open neighborhoods of 0 and 0^*, there is always some point of the form $(x, 0)$ with $x \neq 0$, which is a limit point of both open sets. Thus, 0 and 0^* cannot be separated by closed neighborhoods. Hence, X is not a $T_{2\frac{1}{2}}$-space.

Definition 19.4: (Completely Hausdorff space)

A space X is said to be a *completely Hausdorff space* (or a *functionally Hausdorff space*), if given any two distinct points $x, y \in X$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.

Remark 19.5

Suppose, given $x \neq y \in X$, we have a continuous map $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$. Without loss of generality, assume $f(x) < f(y)$. Consider the function

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} f(x), & t \leq f(x), \\ t, & f(x) \leq t \leq f(y), \\ f(y), & f(y) \leq t. \end{cases} \end{aligned}$$

By the pasting lemma, g is continuous. Then, $h = g \circ f : X \to [f(x), f(y)]$ is a continuous map. By composing with a suitable homeomorphism $[f(x), f(y)] \to [0,1]$, we can then get a continuous map $F : X \to [0,1]$ such that $F(x) = 0$ and $F(y) = 1$.

Exercise 19.6

Suppose Y is a completely T_2 space. Given a space X, suppose for any $x \neq y \in X$, there is a continuous map $f : X \to Y$ such that $f(x) \neq f(y)$. Verify that X is completely T_2. In particular, subspaces and products of completely T_2 spaces are again completely T_2.

Proposition 19.7: (Metric space is completely T_2)

A metrizable space X is completely T_2. Consequently, given a space Y and a continuous injective map $\iota : Y \hookrightarrow X$, we have X is completely T_2. A space which admits a continuous injective map into a metrizable space is called a *submetrizable space*.

Proof

Any metrizable space X is T_2. Thus, we only need to show that it is regular. Suppose d is a metric on X inducing the topology. Then, $\epsilon := d(x,y) \neq 0$. Consider the function,

$$f(z) = d(x,z) + (\epsilon - d(z,y)), \quad z \in X.$$

Since distance function is continuous, it follows that $f : X \to \mathbb{R}$ is a continuous function. Also, $f(y) = 2\epsilon \neq 0 = f(x)$. But then we can get a continuous map $h : X \to [0,1]$ such that $h(x) = 0$ and $h(y) = 1$. Thus, X is completely T_2. □

Proposition 19.8: (Completely T_2-spaces are $T_{2\frac{1}{2}}$)

A completely T_2-space is $T_{2\frac{1}{2}}$.

Proof

Let X be completely T_2. For any distinct $x, y \in X$, get a continuous function $f : X \to [0,1]$ such that $f(x) = 0, f(y) = 1$. Then, consider the closed sets $A := f^{-1}([0, \frac{1}{4}]), B := f^{-1}([\frac{3}{4}, 1])$, which are clearly disjoint. Also, $x \in \underbrace{f^{-1}\left(\left[0, \frac{1}{4}\right)\right)}_{\text{open in } X} \subset A$, and so, $x \in \mathring{A}$. Similarly, $y \in \mathring{B}$. Thus, X is a $T_{2\frac{1}{2}}$-space. □

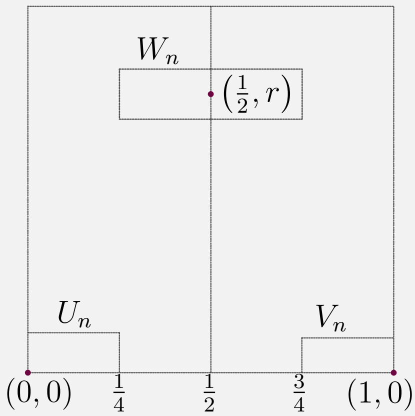
Example 19.9: (Arens square)

Consider $Q := (0,1) \cap \mathbb{Q}$, and let $Q = \sqcup_{q \in Q} Q_q$ be a disjoint union of dense subsets $Q_q \subset Q$, indexed by $q \in Q$. As an explicit example, index each prime number as $\{p_q \mid q \in Q\}$, and then consider

$$Q_q = \left\{ \frac{a}{p_q^i} \;\middle|\; 1 \leq a \leq p_q^i,\ \text{gcd}(a, p_q) = 1,\ i \geq 1 \right\}.$$

Clearly, Q_q is dense in Q, and they are disjoint. Now, consider $A = Q \setminus \bigcup_{q \in Q} Q_q$. Just modify, say, $Q'_{\frac{1}{2}} = Q_{\frac{1}{2}} \cup A$. We still have disjoint dense sets.
Let us now consider the set

$$X = \{(0,0), (1,0)\} \cup \bigcup_{q \in Q} \{q\} \times Q_q \subset \mathbb{R}^2$$



Let us topologize X by declaring basic open neighborhoods for each point.

- For $(0,0)$, declare basic open neighborhoods as the collection

$$U_n := \{(0,0)\} \cup \left\{ (x,y) \in X \;\middle|\; 0 < x < \frac{1}{4},\ 0 < y < \frac{1}{n} \right\}, \quad n \geq 1$$

- For $(1,0)$, declare basic open neighborhoods as the collection

$$V_n := \{(1,0)\} \cup \left\{ (x,y) \in X \;\middle|\; \frac{3}{4} < x < 1,\ 0 < y < \frac{1}{n} \right\}, \quad n \geq 1$$

- For any $\left(\frac{1}{2}, r\right) \in \frac{1}{2} \times Q_{\frac{1}{2}}$, , declare basic open neighborhoods as the collection

$$W_n(r) := \left\{ (x,y) \;\middle|\; \frac{1}{4} < x < \frac{3}{4},\ |y - r| < \frac{1}{n} \right\}, \quad n \geq 1.$$

- Let $X \setminus \{(0,0), (1,0)\} \cup \left\{\frac{1}{2}\right\} \times Q_{\frac{1}{2}}$ inherit the usual subspace topology from $\mathbb{R}^2$.

These neighborhoods form a basis for a topology on X. This space is called the *Arens square*.

Proposition 19.10: ($T_{2\frac{1}{2}} \not\Rightarrow$ Completly T_2 : Arens square space)

Arens square is $T_{2\frac{1}{2}}$-space, but not completely T_2.

Proof

Let us consider the points $a = (0,0)$ and some $b = \left(\frac{1}{2}, r\right)$. Fix some $m, n \geq 1$ such $0 < \frac{2}{m} < r - \frac{1}{n} < r + \frac{1}{n} < 1$. Then, it is easy to see that $\overline{U_m} \cap \overline{W_n} = \emptyset$. Similar argument can be applied to b and $a' = (1,0)$. For any point $c = (q,s)$ with $q \neq \frac{1}{2}$, observe that the y-coordinate s cannot be repeated as $\left(\frac{1}{2}, s\right)$, since we started with a disjoint partition. Thus, using the denseness, we can again get some closed neighborhoods. Hence, the Arens square is a $T_{2\frac{1}{2}}$-space.
Let us show that it is not completely T_2. If possible, suppose $f : X \to [0,1]$ is a continuous map, where X is the Arens square, such that $f(0,0) = 0$ and $f(1,0) = 1$. Since f is continuous, we must have some $m, n \geq 1$ such that

$$(0,0) \in U_n \subset f^{-1}\left[0, \frac{1}{4}\right), \quad (1,0) \in V_m \subset f^{-1}\left(\frac{3}{4}, 1\right].$$

Let us fix some $r \in Q_{\frac{1}{2}}$, with $r < \min\left\{\frac{1}{n}, \frac{1}{m}\right\}$. This is possible since $Q_{\frac{1}{2}}$ is dense in Q. Now, $f\left(\frac{1}{2}, r\right)$ cannot be in both $\left[0, \frac{1}{4}\right)$ and $\left(\frac{3}{4}, 1\right]$. Without loss of generality, we can assume that exists some open interval $U \subset [0,1]$ such that

$$f\left(\frac{1}{2}, r\right) \in U, \quad \left[0, \frac{1}{4}\right] \cap \bar{U} = \emptyset.$$

Then, the pre-images $f^{-1}\left[0, \frac{1}{4}\right]$ and $f^{-1}\bar{U}$ are disjoint closed neighborhoods of $(0,0)$ and $\left(\frac{1}{2}, r\right)$ respectively. Now, $U_n \subset f^{-1}\left[0, \frac{1}{4}\right) \subset f^{-1}\left[0, \frac{1}{4}\right]$. Since $r < \frac{1}{n}$, it follows (Check!) that $\overline{U_n} \cap \overline{W_k} \neq \emptyset$ for any $k \geq 1$. This contradicts $f^{-1}\left[0, \frac{1}{4}\right] \cap \bar{U} = \emptyset$. Hence, the Arens quare is not completely T_2. $\square$

Remark 19.11: (Totally disconnected spaces may not be completely T_2)

It is easy to see that $\mathbb{Q}$, which is a totally disconnected set, is completetly T_2. Indeed, for any $r, s \in \mathbb{Q}$, with $r < s$, get some irrational $r < x < s$. Then,

$$f(t) = \begin{cases} 0, & t < x \\ 1, & x < t, \end{cases}$$

is a continuous function, with $f(r) = 0, f(s) = 1$. But in general, a totally disconnected space need not be completely T_2.
Indeed, we have seen that the Arens square X is not completely T_2. Let us show that it is totally disconnected. Firstly, observe that the second component projection $\pi : X \to [0,1] \cap \mathbb{Q}$ is a continuous map (but the first component projection is not continuous). Now, any two points of X cannot share the same second component, and thus π is injective. Hence, if a connected set $A \subset X$ contains more than one point, $\pi(A)$ will be a connected set of $[0,1] \cap \mathbb{Q}$, with more than one point, a contradiction. Thus, X is totally disconnected.