Topology Course Notes (KSM1C03)

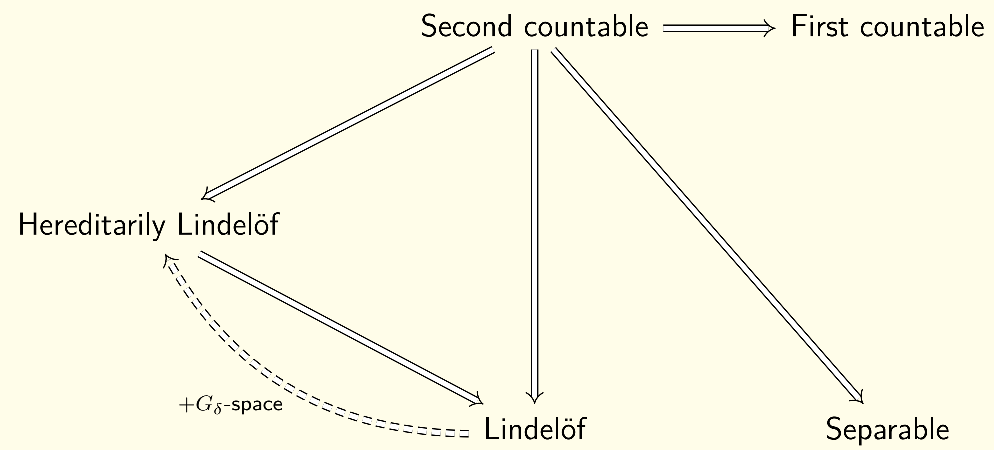
Day 18 : 17$^{\text{th}}$ October, 2025

`countability axioms in metric space -- Lebesgue number lemma`

18.1 Countability axioms in metric spaces

Remark 18.1

We have the implications



Recall, a space is called a G_δ-space if every closed set can be written as the intersection of countably many open sets.

Example 18.2: (Lindelöf is not separable)

Consider an uncountable space X, and fix a point $x_0 \in X$. Let $\mathcal{T}$ be the excluded point topology on X : a proper subset $U \subsetneq X$ is open if and only if $x_0 \notin U$. Then, the only open set containing x_0 is X itself, and hence, X is Lindelöf (in fact, compact). On the other hand, it cannot be separable : for any set $A \subset X$, one can see that $\bar{A} = A \cup \{p\}$. Thus, there cannot be a countable dense subset.

Example 18.3: (Separable is not Lindelöf)

Consider an uncountable space X, and fix a point $x_0 \in X$. Let $\mathcal{T}$ be the particular point topology on X based at x_0 : a nonempty set is open if and only if it contains x_0. Then, $(X, \mathcal{T})$ is separable, as the singleton $\{x_0\}$ is dense in X. But $(X, \mathcal{T})$ is not Lindelöf, as the open cover $\{\{x_0, x\} \mid x \in X\}$ does not have any countable sub-cover.

Theorem 18.4: (Metric space and countability axioms)

Suppose (X, d) is a metric space. Then, X is first countable. Moreover, the following are equivalent.

a) X is second countable.

b) X is separable.

c) X is Lindelöf.

Proof

Given any $x \in X$, consider the open balls $B_n := B_d\left(x, \frac{1}{n}\right)$. It is easy to see that $\{B_n\}$ is a countable basis at x. Thus, X is first countable.
Since any second countable space is separable and Lindelöf, clearly a) $\Rightarrow$ b) and a) $\Rightarrow$ c) holds.
Let us assume X is separable. Then, we have a countable subset $A \subset X$ which is dense in X. Consider the collection

$$\mathcal{B} := \left\{ B_d\left(a, \frac{1}{n}\right) \;\middle|\; a \in A,\ n \geq 1 \right\},$$

which is clearly a countable collection. Let us show that $\mathcal{B}$ is a basis for the topology on (X, d). Suppose $x \in X$, and pick some arbitrary open neighborhood $x \in U \subset X$. Then, for some $n \geq 1$, we have

$$x \in B_d\left(x, \frac{1}{2n}\right) \subset B_d\left(x, \frac{1}{n}\right) \subset U.$$

Since A is dense, we have some $a \in A \cap B_d\left(x, \frac{1}{2n}\right)$. Then, for any $y \in B_d\left(a, \frac{1}{2n}\right)$, we have

$$d(x, y) \leq d(x, a) + d(a, y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \Rightarrow y \in B_d\left(x, \frac{1}{n}\right) \subset U.$$

Thus, $B_d\left(a, \frac{1}{2n}\right) \subset U$. Also, $d(x, a) \leq \frac{1}{2n}$ and so, $x \in B_d\left(a, \frac{1}{2n}\right)$. Thus, $\mathcal{B}$ is a basis, showing b) $\Rightarrow$ a).
Now, suppose X is Lindelöf. For each $n \geq 1$, consider the collection

$$\mathcal{U}_n := \left\{ B_d\left(x, \frac{1}{n}\right) \;\middle|\; x \in X \right\},$$

which is clearly an open cover of X. Hence, there is a countable subcover $\mathcal{V}_n \subset \mathcal{U}_n$. Consider the collection $\mathcal{V} = \bigcup_{n \geq 1} \mathcal{V}_n$, which is clearly a countable collection of open sets. Let us show that $\mathcal{V}$ is a basis for the topology on (X, d). Fix some $x \in X$, and some open neighborhood $x \in U \subset X$. Then, for some $n \geq 1$ we have $x \in B_d\left(x, \frac{1}{2n}\right) \subset B_d\left(x, \frac{1}{n}\right) \subset U$. Since $\mathcal{V}_{2n}$ is a cover, there is some $a \in X$ such that $B_d\left(a, \frac{1}{2n}\right) \in \mathcal{V}_{2n}$ and $x \in B_d\left(a, \frac{1}{2n}\right)$. Now, for any $y \in B_d\left(a, \frac{1}{2n}\right)$, we have

$$d(x, y) \leq d(x, a) + d(a, y) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} \Rightarrow y \in B_d\left(x, \frac{1}{n}\right) \subset U.$$

Thus, $x \in B_d\left(x, \frac{1}{2n}\right) \subset U$. This shows that $\mathcal{V}$ is a basis, proving c) $\Rightarrow$ a). $\square$

Proposition 18.5: (Compact in metric space)

A compact subset of a metric space is closed and bounded.

Proof

Let (X, d) be a metric space, and $C \subset X$ is a compact subset. Since metric spaces are T_2, clearly any compact subset is closed. For any $x_0 \in C$ fixed, consider the open covering $C \subset \bigcup_{n\geq 1} B_d(x_0, n)$. This admits a finite subcover, say, $C \subset \bigcup_{i=1}^{k} B_d(x_0, n_i)$. Taking $n_0 := \max_{1\leq i\leq k} n_i$, we have $C \subset B_d(x_0, n_0)$. Thus, C is bounded. □

Example 18.6: (Closed bounded set in metric space)

In an infinite space X, consider the metric

$$d(x,y) := \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

The induced topology is discrete, and hence, X is not compact. But clearly X is closed in itself, and bounded as $X \subset B_d(x_0, 2)$.

Lemma 18.7: (Lebesgue number lemma)

Suppose (X, d) is a compact metric space, $f : X \to Y$ is a continuous map. Let $\mathcal{V} = \{V_\alpha\}$ be an open cover of $f(X)$. Then, there exists a $\delta > 0$ (called the *Lebesgue number of the covering*) such that for any set $A \subset X$, we have

$$\mathrm{Diam}(A) := \sup_{x,y\in A} d(x,y) < \delta \Rightarrow f(A) \subset V_\alpha \text{for some } \alpha.$$

Proof

For each $x \in X$, clearly, $f(x) \in V_{\alpha_x}$ for some α_x. By continuity of f, we have some $\delta_x > 0$ such that the ball $x \in B_d(x, \delta_x) \subset f^{-1}(V_{\alpha_x})$. Now, $X = \bigcup_{x\in X} B_d\left(x, \frac{\delta_x}{2}\right)$ has a finite subcover, say, $X = \bigcup_{i=1}^{n} B_d\left(x_i, \frac{\delta_{x_i}}{2}\right)$. Set

$$\delta := \min_{1\leq i\leq n} \frac{\delta_{x_i}}{4}.$$

We claim that δ is a Lebesgue number for the covering. Let $A \subset X$ be a set with $\mathrm{Diam}(A) < \delta$. For some $a \in A$, there exists $1 \leq i_0 \leq n$, such that $a \in B_d\left(x_{i_0}, \frac{\delta_{x_{i_0}}}{2}\right)$. Now, for any $b \in A$, we have $d(a, b) \leq \mathrm{Diam}(A) < \delta$. Then,

$$d(x_{i_0}, b) \leq d(x_{i_0}, a) + d(a, b) < \frac{\delta_{x_{i_0}}}{2} + \delta \leq \frac{\delta_{x_{i_0}}}{2} + \frac{\delta_{x_{i_0}}}{4} = \frac{3\delta_{x_{i_0}}}{4} < \delta_{x_{i_0}}.$$

Thus, $A \subset B_d(x_{i_0}, \delta_{x_{i_0}}) \Rightarrow f(A) \subset f\left(B_d\left(x_{i_0}, \delta_{x_{i_0}}\right)\right) \subset V_{\alpha_{x_{i_0}}}$. □