Topology Course Notes (KSM1C03)

Day 17 : 16th October, 2025

`properties of Lindelöf spaces -- separable spaces`

17.1 Properties of Lindelöf spaces

Proposition 17.1: (Image of Lindelöf spaces)

A continuous image of a Lindelöf space is again Lindelöf

Proof

Suppose $f : X \to Y$ is a continuous surjection, and X is Lindelöf. Consider an open cover $Y = \bigcup_\alpha V_\alpha$. Then, we have an open cover $X = \bigcup_\alpha f^{-1}(U_\alpha)$, which admits a countable sub-cover, $X = \bigcup_{i=1}^\infty f^{-1}(U_{\alpha_i})$. Then, $Y = f(X) = \bigcup_{i=1}^\infty U_{\alpha_i}$. Thus, Y is Lindelöf. □

Lindelöf spaces are not well-behaved when considering product or subspaces.

Example 17.2: ($\mathbb{R}_\ell$ is Lindelöf)

Let us show that the lower limit topology $\mathbb{R}_\ell$ on $\mathbb{R}$ is Lindelöf. Suppose $\{U_\alpha\}$ is an open cover. For each x, we have $[x, r_x) \subset U_{\alpha_x}$, for some $r_x \in \mathbb{Q}$. Clearly, $\mathbb{R}_\ell = \bigcup_x [x, r_x)$. Let us consider the space $C = \bigcup_x (x, r_x)$. We claim that $\mathbb{R} \setminus C$ is countable. Indeed, for each $u, v \in \mathbb{R} \setminus C$, with $u < v$, we must have $r_u < r_v$, since otherwise we get $u < v < r_v \le r_u$ and then, $v \in (u, r_u) \subset C$ a contradiction. Thus, we have an injective map

$$\begin{aligned} \mathbb{R} \setminus C &\to \mathbb{Q} \\ u &\mapsto r_u. \end{aligned}$$

But then $\mathbb{R} \setminus C$ is countable, as $\mathbb{Q}$ is countable. Say, $\mathbb{R} \setminus C = \{u_i\}_{i=1}^\infty$. On the other hand, considering $C = \bigcup_{x \in \mathbb{R}} (x, r_x)$ as a collection of open sets in the usual topology of $\mathbb{R}$, we have a countable subcover $C = \bigcup_{i=1}^\infty (x_i, r_{x_i})$. Thus, we have a countable cover,

$$\mathbb{R}_\ell = \bigcup_{i=1}^\infty [u_i, r_{u_i}) \cup \bigcup_{i=1}^\infty [x_i, r_{x_i}) \subset \bigcup U_{\alpha_{u_i}} \cup \bigcup U_{\alpha_{x_i}}.$$

Hence, $\mathbb{R}_\ell$ is Lindelöf.

Example 17.3: ($\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelöf)

Let us now show that the product $X = \mathbb{R}_\ell \times \mathbb{R}_I$ (also known as *Sorgenfrey plane*) is not Lindelöf. Consider the subset $A = \{(x, -x) \mid x \in \mathbb{R}\} \subset X$. It is easy to see that A is open. Next, for each $x \in \mathbb{R}$, consider the open set $U_x = [x, x+1) \times [-x, -x+1) \subset X$. It follows that $A \cap U_x = \{(x, -x)\}$. Now, consider the open cover

$$X = (X \setminus A) \cup \bigcup_{x \in \mathbb{R}} U_x.$$

This cannot have a countable subcover, since A is uncountable.

Definition 17.4: (Hereditarily Lindelöf)

A space X is called *hereditarily Lindelöf* if every subspace $A \subset X$ is Lindelöf.

Proposition 17.5: (Hereditarily Lindelöf if and only if open subsets are Lindelöf)

A space X is hereditarily Lindelöf if and only if every open subspace $U \subset X$ is Lindelöf.

Proof

One direction is trivial. So, suppose every open subspace of X is Lindelöf. Consider an arbitrary subset $A \subset X$, with the subspace topology. Suppose, we have an open cover $A = \bigcup_\alpha U_\alpha$, where $U_\alpha = A \cap V_\alpha$ for $V_\alpha \subset X$ open. Now, $U = \bigcup_\alpha V_\alpha$ is a open cover, which admits a countable subcover, say $U = \bigcup_{i=1}^\infty V_{\alpha_i}$. But then, $A = A \cap U = \bigcup_{i=1}^\infty A \cap V_{\alpha_i} = \bigcup_{i=1}^\infty U_{\alpha_i}$. Thus, A is Lindelöf. Since A was arbitrary, we have X is hereditarily Lindelöf. □

Example 17.6: ($\bar{S}_\Omega$ is not hereditarily Lindelöf)

Recall the space $X = \bar{S}_\Omega = S_\Omega \cup \{\Omega\}$, which was shown to be compact, and hence, Lindelöf. Now, for each $a \in S_\Omega$, consider the open sets $U_a = (a, a+2) = \{a+1\}$. Since S_Ω is uncountable, we have the uncountable discrete space $A = \bigcup_{a \in S_\Omega} (a, a+2) = \bigcup_{a \in S_\Omega} \{a+1\}$. Clearly, this is not Lindelöf. Thus, $\bar{S}_\Omega$ is not hereditarily Lindelöf.

17.2 Separable space

Definition 17.7: (Separability)

Given $A \subset X$, we say A is *dense* in X if $X = \bar{A}$. A space X is called *separable* if there exists a countable dense subset.

Exercise 17.8: (Dense set and open set)

Show that $A \subset X$ is dense if and only for any nonempty open set $U \subset X$ we have $U \cap A \neq \emptyset$.

Exercise 17.9: (Second countablity and seperability)

Show that a second countable space is separable. Check that $\mathbb{R}$ with the cofinite topology is separable, but not second countable.

Proposition 17.10: (Image of separable space)

Let $f : X \to Y$ be countinuous surjection. If X is separable, then so is Y.

Proof

Suppose $A \subset X$ is a countable dense subset. Since f is continuous, we have, $f(\bar{A}) \subset \overline{f(A)} \Rightarrow \overline{f(A)} \supset f(X) = Y \Rightarrow \overline{f(A)} = Y$. Thus, $f(A)$ is dense in Y, which is clearly countable. Hence, Y is separable. □

Proposition 17.11: (Product of separable spaces)

Suppose $\{X_\alpha\}_{\alpha \in I}$ is a countable collection of separable spaces. Then, the product $X = \Pi X_\alpha$ is separable.

Proof

Fix countable dense subsets $A_\alpha \subset X_\alpha$. Fix some $a_\alpha \in A_\alpha$. Then, consider the collection

$$A = \{(x_\alpha) \in \Pi A_\alpha \mid x_\alpha = a_\alpha \text{ for all but finitely many } \alpha \in I\}.$$

By construction, A is countable. Let us show that A is dense in X. Let $U \subset X$ be a basic open sets. Then, $U = \Pi_\alpha U_\alpha$, where $U_\alpha = X_\alpha$ for all $\alpha \in I \setminus \{\alpha_1, \ldots, \alpha_k\}$. Since $X_\alpha = \overline{A_\alpha}$, we have $b_{\alpha_i} \in U_{\alpha_i} \cap A_{\alpha_i}$ for $i = 1, \ldots, k$. Set $b_\alpha = a_\alpha$ for all $\alpha \in I \setminus \{\alpha_1, \ldots, \alpha_k\}$. Then, clearly $b \in U \cap A$. Thus, $\bar{A} = X$. Hence, X is separable. □

Example 17.12: (Subspaces of separable space)

Subspaces of a separable space need not be separable! Consider an uncountable set X, and fix a point $x_0 \in X$. Equip X with the particular point topology based at x_0 (i.e, a nonempty set is open in X if and only if it contains x_0). Then, $\{x_0\}$ is dense in X, and thus X is separable. On the other hand, the set $X \setminus \{x_0\}$ is an uncountable discrete subspace, and hence, cannot be separable.

Definition 17.13: (Nowhere dense subset)

A subset $A \subset X$ is called *nowhere dense* if $\operatorname{int}(\bar{A}) = \emptyset$.

Example 17.14

$\mathbb{Z} \subset \mathbb{R}$ is nowhere dense, and so is the Cantor set (which is uncountable). If X has discrete topology, no subset $A \subset X$ is nowhere dense. The set $A := \mathbb{Z} \cup ((0,1) \cap \mathbb{Q}) \subset \mathbb{R}$ is not nowhere dense.

Exercise 17.15: (Nowhere dense discrete subspace of $\mathbb{R}$)

Show that any discrete subspace $A \subset \mathbb{R}$ is nowhere dense. In particular, $\left\{\frac{1}{n} \,\middle|\, n \geq 1\right\}$ is nowhere dense.

Theorem 17.16: (Nowhere dense equivalence)

Let $A \subset X$ is given. The following are equivalent.

a) $\text{int}(\bar{A}) = \emptyset$.

b) For any nonempty open set $\emptyset \neq U subset X$, we have $A \cap U$ is not dense in U (in the subspace topology).

c) $X \setminus \bar{A}$ is dense in X.

Proof

Suppose $\text{int}(\bar{A}) = \emptyset$. Fix some $\emptyset \neq U \subset X$ open set. Then, $U \not\subset \bar{A}$. Pick some $y \in U \setminus \bar{A}$. Since $\bar{A}$ is closed, we have $V := U \setminus \bar{A}$ is open in X, and hence, open in U as well. Now, clearly $V \cap (U \cap A) = \emptyset$, and hence, $y \notin \overline{U \cap A}^U$. Thus, $U \cap A$ is not dense in U.
Conversely, suppose $A \cap U$ is not dense in U for any nonempty open set $U \subset X$. If possible, suppose $\text{int}(\bar{A}) \neq \emptyset$. Then, there exists some nonempty open set $U \subset \bar{A}$. Pick $y \in U$ and some arbitrary open neighborhood $y \in V \subset U$. Since U is open in X, we have V is open in X as well. Now, $V \subset U \subset \bar{A} \Rightarrow V \cap A \neq \emptyset$ (since $V \cap A = \emptyset \Rightarrow V \cap \bar{A} = \emptyset$ for V open). Thus, we have $\emptyset \neq V \cap A = (V \cap U) \cap A = V \cap (U \cap A)$. Since V was an arbitrary open neighborhood of y in U, we have y is an adherent point of $U \cap A$ (in the subspace topology). Thus, we have $\overline{A \cap U}^U = U$, a contradiction. Hence, $\text{int}(\bar{A}) = \emptyset$.
Let us now assume that $X \setminus \bar{A}$ is dense in X. Then, for any nonempty open set $U \subset X$, we must have $U \cap (X \setminus \bar{A}) \neq \emptyset \Rightarrow U \not\subset \bar{A}$. But then, $\text{int}(\bar{A}) = \emptyset$. Conversely, suppose $\text{int}(\bar{A}) = \emptyset$. Then, for any nonempty open set $U \subset X$, we have $U \not\subset \bar{A} \Rightarrow U \cap (X \setminus \bar{A})$. But this means $X \setminus \bar{A}$ is dense in X. $\square$