Class Test 4

3rd October, 2025

Solutions

Attempt **one** of the questions.

Q1. Let $f : X \to Y$ be a continuous map. Suppose,

- f is closed, i.e, for any closed set $C \subset X$, the image $f(C)$ is closed in Y, and
- f has compact fiber, i.e, for any $y \in Y$ the pre-image $f^{-1}(y)$ is compact in X.

Show that f is a proper map, i.e, for any compact set $K \subset Y$, show that the pre-image $f^{-1}(K)$ is compact in X.

Proof: Let $K \subset Y$ be a compact space. Consider an open cover $f^{-1}(K) \subset \bigcup_{\alpha \in I} U_\alpha$. For any $y \in K$, we have $f^{-1}(y)$ is compact. Hence, we have a finite subset $J_y \subset I$ such that

$$f^{-1}(y) \subset \bigcup_{\alpha \in J_y} U_\alpha.$$

Now, $C_y \coloneqq X \setminus \bigcup_{\alpha \in J_y} U_\alpha$ is closed in X. Hence, $f(C_y)$ is closed in Y, and so, we have an open set,

$$V_y \coloneqq Y \setminus f(C_y) = Y \setminus f\left(X \setminus \bigcup_{\alpha \in J_y} U_y\right).$$

Note that for any $x \in f^{-1}(V_y)$ we have

$$f(x) \in V_y \Rightarrow f(x) \notin f\left(X \setminus \bigcup_{\alpha \in J_y} U_y\right) \Rightarrow x \notin X \setminus \bigcup_{\alpha \in J_y} U_y \Rightarrow x \in \bigcup_{\alpha \in J_y} U_y.$$

Thus, we get $f^{-1}(V_y) \subset \bigcup_{\alpha \in J_y} U_y$. Next, observe that

$$f^{-1}(y) \subset \bigcup_{\alpha \in J_y} U_\alpha \Rightarrow f^{-1}(y) \cap \left(X \setminus \bigcup_{\alpha \in J_y} U_\alpha\right) = \emptyset \Rightarrow y \notin f\left(C_y\right) \Rightarrow y \in V_y.$$

Hence, we have an open cover $K \subset \bigcup_{y \in K} V_y$. Since K is compact, we have a finite sub-cover, $K \subset \bigcup_{i=1}^n V_{y_i}$. It follows that

$$K \subset \bigcup_{i=1}^n V_{y_i} \Rightarrow f^{-1}(K) \subset \bigcup_{i=1}^n f^{-1}(V_{y_i}) \subset \bigcup_{i=1}^n \bigcup_{\alpha \in J_{y_i}} U_\alpha.$$

Thus, we have a finite sub-cover of $f^{-1}(K)$. As K was an arbitrary compact set, we have f is proper.

Q2. Prove or disprove the following statements.

a) Open subsets of a locally compact space is locally compact.

Proof: Let $U \subset X$ be an open set of a locally compact space X. Say $V \subset U$ is open in the subspace topology, and $x \in V$. Now, V is open in X. Then, there is a compact set C such that $x \in \mathring{C} \subset C \subset V$. Clearly, V is compact in U and $\mathring{V}$ is open in U in the subspace topology. Thus, U is locally compact.

b) Closed subsets of a locally compact space is locally compact.

Proof: Let $C \subset X$ be a closed subset of a locally compact space X. Say $U \subset C$ is open in the subspace topology, and $x \in U$. Then, $U = C \cap V$ for some V open in X. Now, by local compactness, we have some compact set K such that $x \in \mathring{K} \subset K \subset V$. Then, consider $K' = K \cap C$, which is a closed subset of a compact set, and hence, itself compact. Clearly,

$$K' = K \cap C \subset V \cap C = U.$$

Also, $\mathring{K} \cap C$ is an open subset of C, which is contained in $K \cap C = K'$. Thus, we have

$$x \in \mathrm{int}_C K' \subset K' \subset U.$$

Hence, C is locally compact.