Topology Course Notes (KSM1C03)

Day 16 : 26th September, 2025

`locally compact space -- compactification`

16.1 Local compactness

Definition 16.1: (Neighborhood)

Given a space X, a *neighborhood* of a point $x \in X$ is any set $N \subset X$ such that $x \in \mathrm{N} \subset N$.

Definition 16.2: (Locally compact space)

A space X is called *locally compact at $x \in X$* if for any given open nbd $x \in U$, there exists a compact neighborhood $x \in C \subset U$. The space X is called *locally compact* if it is so at every point $x \in X$.

Proposition 16.3: (Locally compact Hausdorff)

Suppose X is a Hausdorff space. Then the following are equivalent.

a) X is locally compact.

b) For any $x \in X$ and any open nbd $x \in U \subset X$, there exists an open nbd $x \in V \subset U \subset X$, such that $\bar{V} \subset U$ and $\bar{V}$ is compact.

c) Every $x \in X$ has a cpt nbd.

Proof

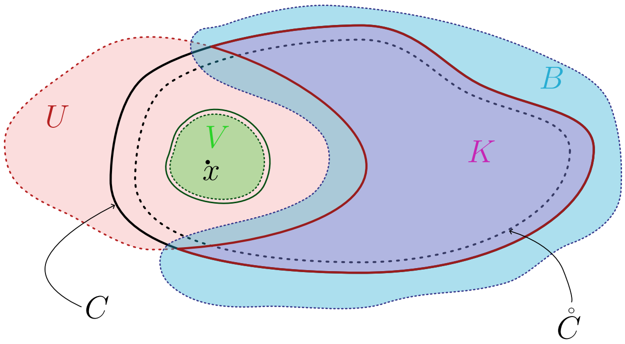
That b) implies local compactness is clear, even without the Hausdorff assumption. Now, suppose X is locally compact, T_2. For an open nbd $x \in U \subset X$, we have some compact nbd $x \in C \subset U$. By the definition of nbd, we have some open nbd $x \in V \subset C \subset U$. Now, since X is T_2, we have C is closed. Hence,

$$V \subset C \Rightarrow \bar{V} \subset \bar{C} = C \subset U.$$

Also, closed subsets of compact is always compact. Thus, $\bar{V}$ is compact. Thus, a) implies b).
Again a) $\Rightarrow$ c) is clear from the definition. Suppose c) holds. Let $x \in U \subset X$ be an open nbd, and $x \in C \subset X$ be a compact nbd. Clearly $x \in W = U \cap \operatorname{int}(C)$ is an open nbd. It follows that $K = C \setminus W$ is a closed subset of the compact set C, and hence, K is compact. Now, $x \notin K$. Since X is T_2, we have open sets $x \in A, K \subset B$, such that $A \cap B = \emptyset$ Set $V = W \cap A = U \cap \operatorname{int}(C) \cap A$,

which is an open nbd $x \in V \subset U$. We observe

$$V \subset W \subset C \Rightarrow \bar{V} \subset \bar{C} = C.$$



Consequently, $\bar{V}$ is compact, being a closed subset of a compact set. Also, $V \subset A$ and $\bar{A} \cap B = \emptyset$ (as $A \cap B = \emptyset$, and B is open). Thus,

$$\bar{V} \subset C \cap (X \setminus B) = C \setminus B = (K \sqcup W) \setminus B = W \setminus B \subset W \subset U.$$

This proves b), and hence a). □

Example 16.4: ($\mathbb{R}$ is locally compact)

Since $\mathbb{R}$ is Hausdorff, it is enough to check that for any $x \in \mathbb{R}$, we have $[x-1, x+1]$ is a compact nbd. Similarly, any $\mathbb{R}^n$ is also locally compact. As for $\mathbb{Q} \subset \mathbb{R}$, for any open set $U = (-\epsilon, \epsilon) \cap \mathbb{Q}$ it follows that $\bar{U} = [-\epsilon, \epsilon] \cap \mathbb{Q}$ is not compact, as it is not sequentially compact. Thus, $\mathbb{Q}$ (which is T_2) is not locally compact.

16.2 Compactification

Definition 16.5: (Compactification)

Given a space X, a *compactification* of X is a continuous injective map $\iota : X \hookrightarrow \hat{X}$, such that $\hat{X} = \overline{\iota(X)}$ is a compact space. We shall identify $X \subset \hat{X}$ as a subspace, and understand $\hat{X}$ as the compactification.

Example 16.6: (Compactification of compact space)

Suppose X is compact. Then $\mathrm{Id} : X \to X$ is trivially a compactification. In fact, if $\hat{X}$ is a Hausdorff compactification of X, then necessarily $\hat{X} = X$ (Check!).

Proposition 16.7: (Alexandroff compactification)

Given any noncompact space $(X, \mathcal{T})$, there exists a compactification $\hat{X} = X \sqcup \{\infty\}$, where ∞ is a point not in X (also denoted as $X^\star$).

Proof

Consider the space $\hat{X} = X \sqcup \{\infty\}$, along with the topology

$$\mathcal{T}_\infty := \mathcal{T} \cup \{\{\infty\} \cup (X \setminus C) \mid C \subset X \text{ is closed and compact}\}.$$

Let us verify that $\mathcal{T}_\infty$ is a topology.

i) $\emptyset \in \mathcal{T} \subset \mathcal{T}_\infty$

ii) $\hat{X} = \{\infty\} \cup (X \setminus \emptyset) \in \mathcal{T}_\infty$, since $\emptyset \subset X$ is a closed, compact subset.

iii) For any $U_\alpha = \{\infty\} \cup (X \setminus C_\alpha)$, where $C_\alpha \subset X$ is closed compact, we have $\bigcup U_\alpha = \{\infty\} \cup (X \setminus \bigcap_\alpha C_\alpha)$. Since arbitrary intersection of closed is closed, and arbitrary intersection of compact is compact, we have $\bigcap_\alpha C_\alpha \subset X$ is closed, compact. Thus, $\bigcup_\alpha U_\alpha \in \mathcal{T}_\infty$. Since finite union of closed (resp. compact) sets are closed (resp. compact), we see that $\bigcap_{i=1} U_i \in \mathcal{T}_\infty$, if $U_i = \{\infty\} \cup (X \setminus C_i)$ for some $C_i \subset X$ closed, compact.

iv) Since $\mathcal{T}$ is a topology, it is closed under arbitrary union and finite intersection.

v) Finally, let us consider some $U \subset X$ open, and some $V = \{\infty\} \cup (X \setminus C)$ for $C \subset X$ closed, compact. We have $U \cap V = U \setminus C$, which is open in X. Also,

$$U \cup V = \{\infty\} \cup (X \setminus C) \cup U = \{\infty\} \cup (X \setminus (C \setminus U)) .$$

Since $C \setminus U$ is a closed subset of a compact set, it is again closed, compact. Thus, $U \cap V \in \mathcal{T}_\infty$.

Thus, $\mathcal{T}_\infty$ is indeed a topology. It is easy to see that the inclusion $\iota : X \hookrightarrow \hat{X}$ is a homeomorphism onto the image (Check!). Also, for ∞, any open neighborhood clearly intersects X, since X itself is not compact. Thus, $\hat{X} = \overline{\iota(X)}$. Finally, let us check that $\hat{X}$ is compact. Indeed, for any open cover $\mathcal{U} = \{U_\alpha\}$, choose some $\infty \in U_{\alpha_0}$. Then, $U_{\alpha_0} = \{\infty\} \cup (X \setminus C)$, where $C \subset X$ is closed and compact. We have $\mathcal{U}$ is an open cover of X, and so, we have a finite subcover, say $C \subset \bigcup_{i=1}^k U_{\alpha_i}$. Then, $\{U_{\alpha_i},\ i = 0, \dots, k\}$ is a finite subcover of $\hat{X}$. □

Remark 16.8: (Alexandroff compactification of compact space)

If X is compact to begin with, then the Alexandroff compactification still produces a compact space $\hat{X} = X \sqcup \{\infty\}$, which contains X as a subspace. But here $\{\infty\}$ is an isolated point, and $\bar{X} = X \subsetneq \hat{X}$. Thus, by our definition, it is not exactly a compactification!

Exercise 16.9: (One-point compactification and Alexandroff compactification)

Consider the space

$$X = \{p, q, x_1, x_2, \dots, y_1, y_2, \dots\} .$$

Give the subspace $\{x_1, x_2, \dots, y_1, y_2, \dots\}$ the discrete topology. For p, declare the open neighborhoods as $\{p\} \cup A$, where $A \subset \{y_1, y_2, \dots\}$ is cofinite. For q, declare the open neighborhoods as $\{q\} \cup B$, where $B \subset \{x_1, x_2, \dots, y_1, y_2, \dots\}$ is cofinite. Check that X is compact with this topology. Now, consider $Y = \{p, x_1, x_2, \dots, y_1, y_2, \dots\}$, which is *noncompact* (Check!). Clearly, $\overline{Y} = X$. Thus, X is a compactification of Y. We claim that X is not the Alexandroff compactification of Y. Indeed, consider the set $K = \{p, y_1, y_2, \dots\} \subset Y$, which is compact (Check!). Also, K is closed in Y. But, $\{q\} \cup (Y \setminus K) = \{q, x_1, x_2, \dots\}$ is not open in X.

Theorem 16.10: (One-point compactification of locally compact Hausdorff space)

Let X be a noncompact space. Then, the one-point compactification $\hat{X}$ is T_2 if and only if X is locally compact, T_2.

Proof

Suppose $\hat{X}$ is T_2. Then, $X \subset \hat{X}$ is clearly T_2. Also, for any $x \in X$, we have open sets $x \in U, \infty \in V$ such that $U \cap V = \emptyset$. Then, $U \subset X$, and $V = \{\infty\} \cup (X \setminus C)$, where $C \subset X$ is a compact (and also closed, as X is T_2). Then, $x \in U \subset C$, that is, C is a compact neighborhood of x. Since X is T_2, it follows that X is locally compact.

Conversely, suppose X is locally compact, T_2. We only need to show that for any $x \in X$, there open sets $x \in U, \infty \in V$ such that $U \cap V = \emptyset$. Since X is T_2, we have an open set $x \in U \subset X$ such that $\bar{U}$ is compact (and hence closed). Then, we have $V = X \setminus \bar{U}$ is an open nbd of ∞ in $\hat{X}$. Clearly, $U \cap V = \emptyset$. Thus, $\hat{X}$ is T_2. $\square$