Topology Course Notes (KSM1C03)

Day 15 : 25th September, 2025

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Zorn's lemma -- well-ordering principle -- ultrafilter lemma
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15.1 A digression : Zorn's Lemma and applications

Definition 15.1: (Partial ordering)

A relation $\leq$ on a set X is called a *partial order* if it satisfies the following.

1. $x \leq x$ for all $x \in X$.
2. $x \leq y, y \leq z \Rightarrow x \leq z$
3. $x \leq y, y \leq x \Rightarrow x = y$

The tuple $(X, \leq)$ is called a *partially ordered set* (or a *poset*). A point $x \in X$ is called a *maximal element* if for any $y \in X$ with $x \leq y$, we have $x = y$.

Definition 15.2: (Chain)

A subset C of a poset $(X, \leq)$ is called a *chain* if C is totally ordered with respect to $\leq$, i.e, for any $c_1, c_2 \in C$, either $c_1 \leq c_2$ or $c_2 \leq c_1$ holds.

Lemma 15.3: (Zorn's lemma)

Given a non-empty poset $(X, \leq)$, suppose every chain has an upper bound in X. Then, X has a maximal element.

Theorem 15.4: (Basis of a vector space)

Given a field $\mathbb{K}$, any non-zero vector space V over $\mathbb{K}$ admits a basis.

Proof

Consider the collection

$$\mathcal{B} := \{B \subset V \mid B \text{ is linearly independent over } \mathbb{K}\}.$$

Note that $\mathcal{B} \neq \emptyset$, since for any $0 \neq v \in V$, we have $B = \{v\} \in \mathcal{B}$. Define

$$B_1 \leq B_2 \Leftrightarrow B_1 \subset B_2, \qquad B_1, B_2 \in \mathcal{B}$$

which is clearly a partial order. Let us consider a chain $\mathcal{C} = \{B_i\}_{i \in I}$ in $(\mathcal{B}, \leq)$. Consider the set $B = \bigcup_{i \in I} B_i$. We check that B is linearly independent. Say, $b_1, \ldots, b_k \in B$. Since $\mathcal{C}$ is a chain, without loss of generality, we have $b_1, \ldots, b_k \in B_{i_0}$ for some $i_0 \in I$. But then clearly $\{b_1, \ldots, b_k\}$ is linearly independent. Hence, $B \in \mathcal{B}$. By construction, we have $B_i \leq B$ for all $i \in I$. Thus, B is an upper bound of $\mathcal{C}$. Then, we have a maximal element, say, $\mathfrak{B} \in \mathcal{B}$. We claim that $\mathfrak{B}$ is a basis of V. If not, then $\mathfrak{B}$ fails to span V. Thus, we must have some

$$v_0 \in V \setminus \mathrm{Span}\langle \mathfrak{B} \rangle .$$

Consider the set $\mathfrak{B}_0 = \mathfrak{B} \sqcup \{v_0\}$. Clearly, $\mathfrak{B}_0$ is linearly independent, and $\mathfrak{B} \subsetneq \mathfrak{B}_0$. Thus contradicts the maximality of $\mathfrak{B}$. Hence, $V = \mathrm{Span}\langle \mathfrak{B} \rangle$. Thus, V admits a basis. □

Theorem 15.5: (Well-ordering principle)

Every nonempty set S admits a well-ordering.

Proof

Consider the collection

$$\mathcal{W} = \{(W, \leq_W) \mid \emptyset \neq W \subset S, \text{ and } \leq_W \text{ is a well-ordering on } W\} .$$

Clearly $\mathcal{W} \neq \emptyset$, since for any $x \in S$, we have the singleton set $\{x\}$ is trivially well-ordered. Let us define $(A, \leq_A) \preceq (B, \leq_B)$ if and only if

i) $A \subset B$,

ii) $\leq_A$ is the restriction of $\leq_B$ (i.e, $a_1 \leq_A a_2$ if and only if $a_1 \leq_B a_2$), and

iii) for any $b \in B \setminus A$ we have $b >_B a$ for all $a \in A$.

It is easy to see that $\preceq$ is a total order on $\mathcal{W}$ (Check!). Suppose $\mathcal{C} = \{(W_\alpha, \leq_\alpha)\}_{\alpha \in I}$ is a chain in $(\mathcal{W}, \preceq)$. Consider

$$W = \bigcup_{\alpha \in I} W_\alpha.$$

Let us define $\leq_W$ as follows. For any $w_1, w_2 \in W$, using the chain condition, we have $w_1, w_2 \in W_{\alpha_0}$ for some $\alpha_0 \in I$. Then, define

$$w_1 \leq_W w_2 \Leftrightarrow w_1 \leq_{\alpha_0} w_2.$$

Again from the chain condition, it follows that $\leq_W$ is well-defined (Check!). Moreover, it is easy to see that $\leq_W$ is a total order (Check!). Let us show that $\leq_W$ is actually a well-order. Say, $\emptyset \neq A \subset W$ is given. Then, $A \cap W_\alpha \neq \emptyset$ for some $\alpha \in I$. Now, $(W_\alpha, \leq_\alpha)$ being a well-order, we have a least element $m_0 = \min A \cap W_\alpha$. We claim that m_0 is the least element of A in the order $\leq_W$. If not, then there is some $a \in A$, with $a <_W m_0$. Now, $a \in W_\beta$ for some $\beta \in I$. From the chain condition, we have two cases.

1. If $W_\beta \leq W_\alpha$, then we have $a \in W_\beta \subset W_\alpha$. But then $a \in W_\alpha \cap A \Rightarrow m_0 \leq_\alpha a \Rightarrow m_0 \leq_W a$, a contradiction.

2. Say, $W_\alpha \leq W_\beta$. We again have two possibilities.

   (a) Say, $a \in W_\beta \setminus W_\alpha$. Then, by the definition of $\preceq$, we have $a \geq_\beta x$ for all $x \in W_\alpha$. In particular, $a \geq_\beta m_0 \Rightarrow a \geq_W m_0$, a contradiction.

(b) Say, $a \in W_\alpha$. But then $m_0 \leq_\alpha a \Rightarrow m_0 \leq_W a$, again a contradiction.

Thus, it follows that $m_0 = \min A$ in the order $\leq_W$. Thus, $(W, \leq_W) \in \mathcal{W}$. Clearly, it is an upper bound of the chain $\mathcal{C}$ (Check!). Now, by Zorn's lemma, there exists a maximal element, say, $(\mathfrak{W}, \leq_{\mathfrak{W}}) \in \mathcal{W}$. We claim that $\mathfrak{W} = S$. If not, then there exists $x \in S \setminus \mathfrak{W}$. Consider

$$\mathfrak{W}_0 = \mathfrak{W} \sqcup \{x\}.$$

Define an order $\leq_0$ on $\mathfrak{W}_0$ by extending the order $\leq_{\mathfrak{W}}$, and declaring $w <_0 x$ for all $w \in \mathfrak{W}$. Then, $(\mathfrak{W}_0, \leq_0)$ is a well-order, which moreover satisfies $(\mathfrak{W}, \leq_{\mathfrak{W}}) \prec (\mathfrak{W}_0, \leq_0)$ (Check!). This violates the maximality. Hence, $\mathfrak{W} = S$, and thus, S admits a well-ordering. □

Theorem 15.6: (Ultrafilter lemma)

A filter $\mathcal{F}$ on a set X is contained in an ultrafilter on X.

Proof

Consider the collection

$$\mathfrak{F} := \{F \mid F \text{ is a filter on } X, \text{ and } \mathcal{F} \subset F.\}$$

Then, $\mathfrak{F} \neq \emptyset$ as $\mathcal{F} \in \mathfrak{F}$. Order $\mathfrak{F}$ by inclusion, i.e, $F_1 \leq F_2$ if and only if $F_1 \subset F_2$. Clearly $(\mathfrak{F}, \leq)$ is a poset. Consider a chain $\mathcal{C} = \{F_i\}_{i \in I}$ in $(\mathfrak{F}, \leq)$. Consider

$$F = \bigcup_{i \in I} F_i.$$

Clearly $\mathcal{F} \subset F$. Let us check that F is a filter on X.

i) Since $\emptyset \notin F_i$ for all $i \in I$, we have $\emptyset \notin F$.

ii) For any $A, B \in F$, by the chain condition, we have $A, B \in F_{i_0}$ for some $i_0 \in I$. But then $A \cap B \in F_{i_0} \Rightarrow A \cap B \in F$.

iii) Say $A \in F$, and $B \supset A$. Now, $A \in F_i$ for some $i \in I$, and then, $B \in F_i \Rightarrow B \in F$.

Thus, F is a filter on X, containing $\mathcal{F}$, and clearly, it is an upper bound of $\mathcal{C}$. Then, by Zorn's lemma, there exists some maximal element, say, $\mathcal{U} \in \mathfrak{F}$. We claim that $\mathcal{U}$ is an ultrafilter on X, which evidently contains $\mathcal{F}$. If not, then there exists some set $S \subset X$ such that

$$S \notin \mathcal{U}, \qquad \text{and} \qquad X \setminus S \notin \mathcal{U}.$$

Then, the collection $\mathcal{U}_0 = \mathcal{U} \cup \{S\}$ has finite intersection property (Check!). But then there is a filter, say, $\mathcal{F}_0 \supset \mathcal{U}_0 \supsetneq \mathcal{U}$, a contradiction to maximality. Hence, $\mathcal{U}$ is an ultrafilter, containing $\mathcal{F}$.□

Here are some more applications, that you can try to do if you want! Or have a look at this note by Keith Conrad.

Exercise 15.7: (Existence of spanning tree)

Using Zorn's lemma, show that every connected (undirected) graph has a spanning tree.

Exercise 15.8: (Existence of maximal ideal)

Let R be a commutative ring with 1. Using Zorn's lemma, show that every ideal $I \subset R$ is contained in a maximal ideal.

Exercise 15.9: (Description of nilradical)

Let R be a commutative ring with 1. Using Zorn's lemma, show that

$$\bigcap_{\mathfrak{p} \subset R \text{ is a prime ideal}} \mathfrak{p} = \{x \in R \mid x^n = 0 \text{ for some } n \geq 1\},$$

which is also known as the *nilradical* of R.