Class Test 3

23rd September, 2025

Name: ____

Time: 40 min **Marks:** ____/10

Q1. Consider the space $X = \{0, 1, 2, \ldots\}$, equipped with the topology

$$\mathcal{T} := \{\emptyset, X\} \cup \{S \mid S \subset \{1, 2, 3, \ldots\}\} \cup \{\{0\} \cup A \mid A \subset \{1, 2, 3, \ldots\} \text{ is cofinite.}\}$$

Prove or disprove the following statements.

a) $(X, \mathcal{T})$ is compact.

b) $(X, \mathcal{T})$ is first countable.

c) $(X, \mathcal{T})$ is second countable.

Show that $(X, \mathcal{T})$ is homeomorphic to $K = \{0\} \cup \{\frac{1}{n} | n \geq 1\} \subset \mathbb{R}$ with the usual topology. $1 \times 3 + 2 = 5$

Q2. Suppose X is a Hausdorff space. Let $B \subset X$ be compact.

a) If $x \in X \setminus B$, then show that there exists open neighborhoods $x \in U$ and $B \subset V$ such that $U \cap V = \emptyset$.

b) If $A \subset X \setminus B$ is a compact set, then show that there exists open neighborhoods $A \subset U$ and $B \subset V$ such that $U \cap V = \emptyset$. $2\frac{1}{2} + 2\frac{1}{2} = 5$