Class Test 3

23rd September, 2025

Solutions

Q1. Consider the space $X = \{0, 1, 2, \dots\}$, equipped with the topology

$$\mathcal{T} := \{\emptyset, X\} \cup \{S \mid S \subset \{1, 2, 3, \dots\}\} \cup \{\{0\} \cup A \mid A \subset \{1, 2, 3, \dots\} \text{ is cofinite.}\}$$

Prove or disprove the following statements.

a) $(X, \mathcal{T})$ is compact.

Proof: Suppose $\{U_\alpha\}$ is a cover of X. Now, $0 \in U_{\alpha_0}$ for some α_0. Then, $U_{\alpha_0} = \{0, 1, 2, 3, \dots\} \setminus \{n_1, \dots, n_k\}$ for some $n_i \geq 1$. Now, $n_i \in U_{\alpha_i}$ for $i = 1, \dots, k$. Then, $\{U_{\alpha_i},\ i = 0, \dots, k\}$ is a finite sub-cover.

b) $(X, \mathcal{T})$ is first countable.

Proof: For any $n \geq 1$, we have $\{n\}$ itself is open, and hence, $\{\{n\}\}$ a countable neighborhood basis. For 0, the collection

$$\{\{0\} \cup A \mid A \subset \{1, 2, 3, \dots\} \text{ is cofinite}\}$$

is also countable, as the collection of finite subsets of $\{1, 2, \dots\}$ is a countable collection. Since these are all the open sets containing 0, clearly it is a neighborhood basis. Alternatively,

$$\{U_n = \{0, n, n+1, \dots\} \mid n \geq 1\}$$

is another countable neighborhood basis at 0.

c) $(X, \mathcal{T})$ is second countable.

Proof: Since X itself is countable, and $(X, \mathcal{T})$ is first countable, it follows that the space is second countable. Indeed, we have a countable basis

$$\{\{n\},\ n \geq 1\} \cup \{\{0\} \cup A \mid A \subset \{1, 2, 3, \dots\} \text{ is cofinite}\}.$$

Show that X is homeomorphic to $K = \{0\} \cup \{\frac{1}{n} | n \geq 1\} \subset \mathbb{R}$ with the usual topology.

Proof: Consider the obvious map, with its inverse

$$\begin{aligned} f : X &\longrightarrow K \\ 0 &\longmapsto 0 \\ n &\longmapsto \frac{1}{n}, \end{aligned} \qquad \begin{aligned} g : K &\longrightarrow X \\ 0 &\longmapsto 0 \\ \frac{1}{n} &\longmapsto n. \end{aligned}$$

Since $g = f^{-1}$, it follows that f is bijective. Let us check that f continuous. Indeed, any $\left\{\frac{1}{n}\right\}$ is discrete in K, and also,

$$f^{-1}\left(\left\{\frac{1}{n}\right\}\right) = \{n\}$$

is open in X. For $0 \in K$, any open neighborhood is of the form

$$U := K \setminus \left\{ \frac{1}{n_1}, \dots, \frac{1}{n_k} \right\},$$

and clearly, $f^{-1}(U) = \{0, 1, 2, \dots\} \setminus \{n_1, \dots, n_k\}$ is open in X. Finally, $f : X \to K$ is a continuous bijection, from a compact space $(X, \mathcal{T})$ to a T_2 space K. Hence, f is an open map. (Alternatively, similar argument shows that g is continuous.) Thus, $f : X \to K$ is a homeomorphism.

Q2. Suppose X is a Hausdorff space. Let $B \subset X$ be compact.

a) If $x \in X \setminus B$, then show that there exists open neighborhoods $x \in U$ and $B \subset V$ such that $U \cap V = \emptyset$.

Proof: Since X is T_2, for each $b \in B$, there exists some open sets $x \in U_b, b \in V_b$ such that $U_b \cap V_b = \emptyset$. Then, we have a cover $B \subset \bigcup_{b \in B} V_b$, which admits a finite sub-cover, say, $B \subset \bigcup_{i=1}^k V_{b_i}$. Consider $U := \bigcap_{i=1}^k U_{b_i}$, and $V := \bigcup_{i=1}^k V_{b_i}$. Then, $x \in U, B \subset V$ are open neighborhoods. Also,

$$U \cap V = \bigcup_{i=1}^{k} U \cap V_{b_i} = \bigcup_{i=1}^{k} (U_{b_1} \cap \dots \cap U_{b_i} \cap \dots \cap U_{b_k}) \cap V_{b_i} = \emptyset.$$

b) If $A \subset X \setminus B$ is a compact set, then show that there exists open neighborhoods $A \subset U$ and $B \subset V$ such that $U \cap V = \emptyset$.

Proof: For each $a \in A$, as $a \in X \setminus B$, by a), we have open neighborhoods $a \in U_a, B \subset V_a$ such that $U_a \cap V_a = \emptyset$. Then, we have a cover $A \subset \bigcup_{a \in A} U_a$, which admits a finite subcover $A \subset \bigcup_{i=1}^k U_{a_i}$. Consider $U := \bigcup_{i=1}^k U_{a_i}$, and $V := \bigcap_{i=1}^k V_{a_i}$. Then, we have open neighborhoods $A \subset U, B \subset V$. Also,

$$U \cap V = \bigcup_{i=1}^{k} U_{a_i} \cap V = \bigcup_{i=1}^{k} U_{a_i} \cap (V_{a_1} \cap \dots \cap V_{a_i} \cap \dots \cap V_{a_k}) = \emptyset.$$