Topology Course Notes (KSM1C03)

Day 14 : 19th September, 2025

`uncountable ordinal -- filter -- ultrafilter lemma -- Tychonoff's theorem`

14.1 Properties of S_Ω

Proposition 14.1: (Properties of S_Ω)

Suppose S_Ω is given the order topology.

a) For any set $A \subset S_\Omega$, the union $\bigcup_{a\in A} S_a$ is either a section (and hence countable), or all of S_Ω.

b) Any countable set of S_Ω is bounded

c) S_Ω is sequentially compact.

d) S_Ω is limit point compact.

e) S_Ω is not compact.

f) S_Ω is first countable.

Proof

a) If A admits an upper bound, then it admits a least upper bound, say, b. We claim that $\bigcup_{a\in A} S_a = S_b$. Indeed, for any $x < a \in A$, we have $x < a \le b$ and so $x \in S_b$. On the other hand, for any $x < b$, we have x is not an upper bound of A, and so, $x < a \le b$ for some $a \in A$. Then, $x \in S_a$.

Otherwise, assume A is not bounded. Suppose $\bigcup_{a\in A} S_a$ is not all of S_Ω. Pick some $b \in S_\Omega \setminus \bigcup_{a\in A} S_a$. Now, b is not an upper bound of A (as A is not upper bounded). So, $b < a \in A$. But then $b \in S_a$, a contradiction.

b) For a countable set $A \subset S_\Omega$, the subset $\bigcup_{a\in A} S_{a+1}$ is countable, and hence, not all of S_Ω. Then, $A \subset \bigcup_{a\in A} S_{a+1} = S_b$ for some b. Clearly, b is an upper bound of A.

c) WLOG, suppose $\{x_n\}$ be a sequence of distinct elements in S_Ω. Consider

$$x_{n_k} = \min \{x_n \mid n \ge k\}.$$

Then, clearly $x_{n_1} < x_{n_2} < \dots$. Now, $\{x_{n_k}\}$ being countable set, is bounded, and hence admits a least upper bound, say b. Clearly $b \notin \{x_{n_k}\}$, as the subsequence is strictly increasing. For any open set $b \in U \subset S_\Omega$, we have $b \in (x, b] \subset U$. Now, x is not an upper bound of $\{x_{n_k}\}$, and hence, $a < x_{n_{k_0}} < b$ for some k_0. But then $a < x_{n_l} < b$ for any $l \geq k_0$. In other words, $x_{n_l} \in U$ for all $l \geq k_0$. Thus, $x_{n_k} \to b$.

d) Since S_Ω is sequentially compact, it is limit point compact.

e) For each $x \in S_\Omega$, consider the open sections $S_{x+1} := \{y \in X \mid y < x+1\}$, which are open. Here $x+1$ is the successor of x. Clearly, $S_\Omega = \bigcup_{x \in S_\Omega} S_{x+1}$. If possible, suppose, there is a finite subcover, $S_\Omega = \bigcup_{i=1}^{n} S_{x_i+1}$. But the right-hand side is a finite union of countable sets, and hence countable, whereas S_Ω is uncountable. This is a contradiction.

f) For any $x \in S_\Omega$, we have the section $S_x = \{a \mid a < x\}$ is countable. Consider the open sets $\{U_a = (a, x+1) \mid a < x\}$, which are all open neighborhoods of x. It is clear that this is a countable basis at x (Check!).

□

Proposition 14.2: ($\bar{S}_\Omega$ is not first countable)

The space $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ is not first countable at Ω.

Proof

Observe that the basic open sets containing Ω are of the form $(x, \Omega]$ for $x \in S_\Omega$. If possible, suppose, there is countable neighborhood basis at Ω, say, $\{U_i\}$. We then have $\Omega \subset (x_i, \Omega] \subset U_i$ for some $x_i \in S_\Omega$. Now, $\bigcup S_{x_i} = S_b$ for some $b \in S_\Omega$. Consider the basic open set $(b+1, \Omega]$. There is some $\Omega \in (x_i, \Omega] \subset U_i \subset (b+1, \Omega]$. But then $b+1 \leq x_i$, a contradiction. Hence, $\bar{S}_\Omega$ is not first countable at Ω. □

14.2 (Ultra)Filters

Definition 14.3: (Filter and ultrafilter)

Given a set X, a *filter* on it is a collection $\mathcal{F} \subset \mathcal{P}(X)$ of subsets such that the following holds.

a) $\emptyset \notin \mathcal{F}$.

b) For any $A, B \subset X$, we have $A \cap B \in \mathcal{F}$ if and only if $A, B \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set X, is called an *ultrafilter* if for any $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Exercise 14.4: (Filter equivalent definition)

Given any collection $\mathcal{F} \subset \mathcal{P}(X)$ of subsets, the following are equivalent.

a) For any $A, B \subset X$, we have $A \cap B \in \mathcal{F}$ if and only if $A, B \in \mathcal{F}$

b) $\mathcal{F}$ satisfies the following.

   i) $\mathcal{F}$ is closed under finite intersection, i.e, $F_1, \ldots, F_n \in \mathcal{F}$ implies $\cap_{i=1}^{n} F_n \in \mathcal{F}$.

   ii) $\mathcal{F}$ is closed under supersets, i.e, if $A \in \mathcal{F}$, then $B \in \mathcal{F}$ whenever $B \supset A$.

Example 14.5: (Principal ultrafilter)

For any $x \in X$ fixed, consider the collection

$$\mathcal{F} = \{A \subset X \mid x \in A\}.$$

It is easy to see that $\mathcal{F}$ is an ultrafilter on X, Such ultrafilters are called the *principal ultrafilter*. Any ultrafilter which is not principal, is called a *free ultrafilter*.

Theorem 14.6: (Ultrafilter lemma)

Every filter on a set X is contained in an ultrafilter.

Proof

Let $\mathcal{F}$ be a filter on X. Consider the collection

$$\mathfrak{F} := \{\mathcal{G} \mid \mathcal{G} \text{ is a filter on } X, \text{ and } \mathcal{F} \subset \mathcal{G}\}.$$

It follows that every chain (ordered by inclusion) in $\mathfrak{F}$ admits a maximal element, given by the union. Then, by Zorn's lemma, $\mathfrak{F}$ admits a maximal element, say, $\overline{\mathcal{F}}$. Since $\overline{\mathcal{F}}$ is a maximl filter, it is an ultrafilter, which contains $\mathcal{F}$ by construction. □

Definition 14.7: (Convergence of filter)

Given a filter $\mathcal{U}$ on a space X, we say $\mathcal{U}$ converges to a point $x \in X$, if for any open neighborhood $x \in U$, we have $U \in \mathcal{U}$.

Theorem 14.8: (Ultrafilter and compactness)

A space X is compact if and only if every ultrafilter on X converges to at least one point.

Proof

Suppose X is a compact space. Let $\mathcal{U}$ be an ultrafilter on X. If possible, suppose $\mathcal{U}$ does not converge to any point in X. Then, for each $x \in X$, there exists an open nbd $x \in U_x$ such that $U_x \notin \mathcal{U}$. Since $\mathcal{U}$ is ultrafilter, this means $X \setminus U_x \in \mathcal{U}$. Now, $X = \bigcup_{x \in X} U_x$ admits a finite sub-cover, say, $X = \bigcup_{i=1}^k U_{x_i}$. This, means

$$\emptyset = X \setminus X = \bigcap_{i=1}^{k} (X \setminus U_{x_i}) \in \mathcal{U},$$

as $\mathcal{U}$ is closed under finite intersection. This is a contradiction as $\emptyset \notin \mathcal{U}$.
Conversely, suppose X is not compact. Then, there exists an open cover, $\mathcal{U} = \{U_\alpha\}$ such that there is no finite sub-cover. Consider the collection

$$\mathcal{F} := \{F_\alpha = X \setminus U_\alpha\}.$$

Note that for any finite collection, we have $\cap_{i=1}^k F_{\alpha_i} = X \setminus \bigcup_{i=1}^k U_{\alpha_i} \neq \emptyset$. In other words, $\mathcal{F}$ has finite intersection property. Then, we can close $\mathcal{F}$ under finite intersections, and then under supersets, to get a filter, say, $\mathfrak{F} \supset \mathcal{F}$. But $\mathfrak{F}$ is contained in some ultrafilter, say $\mathfrak{U} \supset \mathfrak{F}$. Now, for any $x \in X$, we have $X \in U_\alpha$ for some α. Then, $F_\alpha = X \setminus U_\alpha \in \mathfrak{U} \Rightarrow U_\alpha \notin \mathfrak{U}$. Thus, $\mathfrak{U}$ does not converge to any $x \in X$, a contradiction. □

14.3 Tychonoff's Theorem

Theorem 14.9: (Tychonoff's Theorem)

Given a collection $\{X_\alpha\}$ of compact spaces, the product $X = \Pi X_\alpha$, with the product topology, is a compact space.

Proof

Suppose $\mathcal{U}$ is an ultrafilter on X. For the projection map $\pi_\alpha : X \to X_\alpha$, we have the ultrafilter

$$\mathcal{U}_\alpha := (\pi_\alpha)_* \mathcal{U} = \left\{A \subset X_\alpha \;\middle|\; (\pi_\alpha)^{-1}(A) \in \mathcal{U}\right\}$$

on X_α. Since X_α is compact, $\mathcal{U}_\alpha$ converges to some point in X_α. By the axiom of choice, we have some $x = (x_\alpha) \in X$ such that $\mathcal{U}_\alpha$ converges to x_α for each α. Let us show that $\mathcal{U}$ converges to x. Observe that for any open neighborhood $x \in U \subset X$, we have U is generated by the sub-basic open sets of the form $\{\pi_\alpha^{-1}(V) \mid V \subset X_\alpha\}$. Since a filter is closed under finite intersection and supersets, if we are able to show that any sub-basic open neighborhood of x is an element of $\mathcal{U}$, we are done. But for any $V \subset X_\alpha$ open, with $x \in \pi_\alpha^{-1}(V)$ precisely when $x_\alpha \in V$. Since $\mathcal{U}_\alpha$ converges to x_α, we have $V \in \mathcal{U}_\alpha \Rightarrow \pi_\alpha^{-1}(V) \in \mathcal{U}$. Hence, $\mathcal{U}$ converges to x. Since $\mathcal{U}$ is an arbitrary ultrafilter, we have X is compact. □

Proposition 14.10: (Axiom of choice from Tychonoff)

Suppose Tychonoff's theorem is true. Then, axiom of choice holds.

Proof

Let $\{X_\alpha\}$ be an arbitrary collection nonempty sets. Since a set cannot be an element of itself, we have new sets $Y_\alpha = X_\alpha \sqcup \{X_\alpha\}$. For simplicity, denote $p_\alpha = \{X_\alpha\} \in Y_\alpha$. Now, give a topology on Y_α as

$$\mathcal{T}_\alpha = \{\emptyset, \{p_\alpha\}, X_\alpha, Y_\alpha\}$$

. Clearly $(Y_\alpha, \mathcal{T}_\alpha)$ is a compact space, having only finitely many open sets. Consider the product $Y = \Pi_\alpha Y_\alpha$. Now, for each α, we have the sub-basic open set

$$U_\alpha := \{y \in Y \mid \pi_\alpha(y) = p_\alpha\} = \pi_\alpha^{-1}(p_\alpha),$$

since $\{p_\alpha\}$ is open in Y_α. We claim that $\{U_\alpha\}$ has not finite sub-cover. If possible, suppose, $Y = \bigcup_{i=1}^n U_{\alpha_i}$. Then, make finitely many choices : $x_i \in X_{\alpha_i}$, and define x by setting $\pi_\alpha(x) = p_\alpha$ for $\alpha \notin \{a_1, \ldots, \alpha_n\}$ and $\pi_{\alpha_i}(x) = x_i$ for $1 \leq i \leq n$. Then, clearly $x \notin \bigcup_{i=1}^n U_{\alpha_i}$, a contradiction. Thus, the collection $\{U_\alpha\}$ admits no finite sub-cover. By Tychonoff's theorem, Y is compact. Hence, $\{U_\alpha\}$ is not a covering of Y. So, there exists some $y \in Y \setminus \bigcup_\alpha U_\alpha$. Observe that $\pi_\alpha(y) \in X_\alpha$, as $y_\alpha \neq p_\alpha$. Thus, $y \in \Pi X_\alpha$. This is precisely the axiom of choice. □

Proposition 14.11: (Compact but not sequentlly compact)

The product space $X = [0,1]^{[0,1]} = \Pi_{0 \leq t \leq 1}[0,1]$ is compact, but not sequentially compact.

Proof

It follows from Tychonoff's theorem that the product space $X = [0,1]^{[0,1]}$ is compact, since each $[0,1]$ is so. For each $n \geq 1$, consider the function $\alpha_n : [0,1] \to \{0,1\}$ defined by

$$\alpha_n(x) = \text{the } n^{\text{th}} \text{ digit in the binary expansion of } x.$$

Clearly, $\{\alpha_n\}$ is a sequence in X. If possible, suppose, $\alpha_{n_k} \to \alpha \in X$. Then, for each $x \in [0,1]$, we must have $\alpha_{n_k}(x) \to \alpha(x)$. Consider any point x such that $\alpha_{n_k}(x)$ is 0 or 1 according as k is even or odd. Clearly the sequence $\alpha_{n_k}(x)$ cannot converge, a contradiction. Thus, X is not sequentially compact. □