Topology Course Notes (KSM1C03)

Day 13 : 18th September, 2025

`order topology -- compact interval -- well-ordereing -- uncountable ordinal`

13.1 Order topology and compactness

Definition 13.1: (Order topology)

Given any totally ordered set $(X, \leq)$, the *order topology* on X is defined as the topology generated by the subbasis consisting of rays $\{x \in X \mid x < a\}$ and $\{x \in X \mid a < x\}$ for all $a \in X$.

Exercise 13.2: (Order topology basis)

Given a total order $(X, \leq)$ (with at least two points), check that the following collection

$$\mathcal{B} := \{(a, b) \mid a, b \in X,\ a < b\},$$

is a basis for the order topology. Here, the intervals are defined as $(a, b) := \{x \in X \mid a < x < b\}$.

Proposition 13.3: (Order topology is T_2)

Let $(X, \leq)$ be a totally ordered set equipped with the order topology. Then, X is T_2.

Proof

Let $a \neq b \in X$. Without loss of generality, $a < b$. There are two possibilities. Suppose there is some c such that $a < c < b$. Then, consider $U = \{x \in X \mid x < c\}$ and $V = \{x \in X \mid c < x\}$. Clearly, $a \in U, b \in V$ and $U \cap V = \emptyset$. If no such c exists, take $U = \{x \mid x < b\}$ and $V = \{x \mid a < x\}$. ☐

Theorem 13.4: (Compact sets in ordered topology)

Suppose X is a totally ordered space, with the least upper bound property : any upper bounded set $A \subset X$ has a least upper bound. Then, for any $, b \in X$ with $a < b$, the interval $[a, b] = \{c \in X \mid a \leq c \leq b\}$ is compact.

Proof

Suppose $\mathcal{U} = \{U_\alpha\}$ be an open cover of $[a, b]$.

For any $x \in [a, b)$, we first observe that there is some $y \in (x, b]$ such that $[x, y]$ is covered by at most two elements of $\mathcal{U}$. If x has an immediate successor in X, let $y = x + 1$. Then, $y \in (x, b]$, and $[x, y]$ contains exactly two points. Clearly, $[x, y]$ can be covered by at most two open sets of $\mathcal{U}$. If there is no immediate successor, get $x \in U_\alpha$, and some $x < c \leq b$ such that $[x, c) \subset U_\alpha$.

Since x has no immediate successor, we have some $x < y < c$ so that $[x, y] \subset [x, c) \subset U_\alpha$. Now, consider the collection

$$\mathcal{A} := \{c \in [a, b] \mid [a, c] \text{ is covered by finitely many } U_\alpha.\}$$

Observe that for a, we have some $a < y \leq b$ such that $[a, y]$ is covered by at most two open sets of $\mathcal{U}$. Thus, $y \in \mathcal{A}$. Clearly $\mathcal{A}$ is upper bounded by b. Let c be the least upper bound of $\mathcal{A}$. We then have, $a < c \leq b$.

We show that $c \in \mathcal{A}$. We have $c \in U_\alpha$ for some α. Then, there is some c' such that $(c', c] \subset U_\alpha$. Now, being the least upper bound, we must have some $z \in \mathcal{A}$ such that $c' < z \leq c$. Then, $[a, z]$ lies in finitely many opens of $\mathcal{U}$. Adding U_α to that finite collection, we get a finite cover of $[a, c] = [a, z] \cup [z, c]$. Thus, $c \in \mathcal{A}$.

Finally, we claim that $c = b$. If not, then there is some $c < y \leq b$ such that $[c, y]$ is covered by at most two opens from $\mathcal{U}$. This implies that $[a, y] = [a, c] \cup [c, y]$ admits a finite sub-cover, and hence, $y \in \mathcal{A}$. But this contradicts c is an upper bound. Thus, $c = b$.

In other words, $[a, b]$ is covered by finitely many open sets of $\mathcal{U}$. □

Corollary 13.5: (Intervals are compact)

For any real numbers $a < b$, the interval $[a, b]$ is compact in the usual topology of real line.

Proof

It is clear that $\mathbb{R}$ is a totally ordered set, equipped with the order topology. Also, $\mathbb{R}$ has the least upper bound property. Hence, $[a, b]$ is compact. □

13.2 Well-ordering

Definition 13.6: (Well-order)

A *well-ordering* on a set X is a total order, such that every non-empty subset has a least element. Explicitly, it is a relation $\mathcal{R} \subset X \times X$, denote, $a \leq b$ if and only if $(a, b) \in \mathcal{R}$, such that the following hold.

a) **(Reflexivity)** $x \leq x$ for all $x \in X$.

b) **(Transitivity)** If $x \leq y$ and $y \leq z$, then $x \leq z$.

c) **(Totality)** For $x, y \in X$ either $x \leq y$ or $y \leq x$.

d) **(Antisymmetric)** If $x \leq y$ and $y \leq x$, then $x = y$.

e) For any $\emptyset \neq A \subset X$, there exists $a_0 \in A$ such that for all $a \in A$ we have $a_0 \leq a$. We call it *the least element* of A (which is unique, by antisymmetry)

Given a well-ordered set $(X, \leq)$, and a point $x \in X$, the *section* (or *initial segment*) is defined as $S_x := \{y \in X \mid y < x\}$.

Proposition 13.7: (Successor in well-order)

Given a well-ordering $(X, \leq)$, each $x \in X$ (except possibly the greatest element) has an immediate successor, denoted, $x + 1$. That is, $x < x + 1$, and there is no $y \in X$ such that $x < y < x + 1$.

Proof

For any $x \in X$, consider the set

$$U_x := \{y \in X \mid x < y\}.$$

If x is not the greatest element of X, then $U_x \neq \emptyset$, and hence, has a least element. This least element is the successor (Check!). □

Theorem 13.8: (Well-ordering principle)

Every set admits a well-ordering.

Remark 13.9: (Construction of uncountable well-order)

The well-ordering principle (also known as *Zermelo's theorem* named after Ernst Zermelo) is equivalent to the axiom of choice. On the other hand, explicitly constructing an uncountable well-order is possible without using the (full strength of) axiom of choice!

Theorem 13.10: (Construction of an uncountable well-order)

There exists an uncountable well-ordered set.

Proof

Consider $\mathbb{N}$ with the usual order, and observe that any subset $A \subset \mathbb{N}$ is a well-ordering with this ordering. Consider the set

$$\mathcal{A} := \{(A, \prec) \mid A \in \mathcal{P}(\mathbb{N}), \prec \text{ is a strict well-order on } A\}.$$

Since $\mathcal{P}(\mathbb{N})$ is uncountable, and since every subset admits at least one well-order, clearly, $\mathcal{A}$ is uncountable. Let us define a relation

$$(A, \prec_A) \sim (B, \prec_B) \Leftrightarrow ((A, \prec_A)) \text{ is order-isomorphic to } (B, \prec_B).$$

Then, $\sim$ is an equivalence relation on $\mathcal{A}$ (check!). On the equivalence classes, define a new relation

$$[A, \prec_A] \ll [B, \prec_B] \Leftrightarrow (A, \prec_A) \text{ is order-isomorphic to some section of } (B, \prec_B).$$

Then, $\ll$ is a well-defined (strict) well-ordering on $\Omega := \mathcal{A}/_\sim$ (Check! (It is tricky!)). □

Proposition 13.11: (Construction of S_Ω)

There exists a well-ordering, denoted S_Ω (or, ω_1, known as the *first uncountable ordinal*), such that

i) S_Ω is uncountable, and

ii) for each $x \in S_\Omega$ the section $S_x := \{y \in S_\Omega \mid y < x\}$ is countable.

Proof

Suppose $(A, \leq)$ is an uncountable well-ordered set. Then, on $B = A \times \{0, 1\}$, the dictionary order is again a well-ordering (check!). Observe that for any $x = (a, 1)$, the section $S_x = \{y \in B \mid y < x\}$ is uncountable. Consider the set

$$S := \{x \in B \mid S_x \text{ is uncountable}\}.$$

This is non-empty, and hence, admits a least element $\Omega \in S$. Denote

$$S_\Omega := \{x \in B \mid x < \Omega\}.$$

Clearly S_Ω itself is uncountable, as $\Omega \in S$. But that for any $x \in S_\Omega$, we have the section S_x is countable. Since S_Ω is a section of a well-ordering, it is itself well-ordered (check!). $\square$

We shall denote

$$\bar{S}_\Omega := S_\Omega \cup \{\Omega\},$$

and give it the obvious ordering : for any $x \in S_\Omega$ set $x < \Omega$. Note that S_Ω is a section in $\bar{S}_\Omega$, so that the notation is consistent.

Theorem 13.12: ($\bar{S}_\Omega$ is compact)

The space $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ is compact.

Proof

Let m_0 be the least element of S_Ω. On $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$, extend the ordering by setting $x < \Omega$ for all $x \in S_\Omega$. Observe that this is a total order. And moreover, $\bar{S}_\Omega = [m_0, \Omega]$ is a closed interval. Let us check the least upper bound property. Say $A \subset \bar{S}_\Omega$. If $\Omega \in A$, then clearly, Ω is the least upper bound of A. WLOG, assume $\Omega \notin A$, that is, $A \subset S_\Omega$. We have two possibilities. If A is bounded in S_Ω, consider the set

$$X = \{b \in S_\Omega \mid b \text{ is an upper bound of } A\}.$$

As X is nonempty, there exists a least element, say, $b_0 \in X$. By definition, it is the least upper bound of A. Suppose A is unbounded in S_Ω. Clearly, Ω is an upper bound of A. We claim that Ω is the least upper bound. If not, then there is some upper bound $x < \Omega$, which implies A is bounded by $x \in S_\Omega$, a contradiction. Thus, $\bar{S}_\Omega$ has the least upper bound property. So, $\bar{S}_\Omega$ is compact. $\square$