Topology Course Notes (KSM1C03)

Day 12 : 17^{th} September, 2025

`sequential compactness -- limit point compactness -- second countable -- Lindelöf`

12.1 Sequential Compactness (Cont.)

Definition 12.1: (Countably compact)

A space X is called *countably compact* if every countable open cover admits a finite sub-cover.

Proposition 12.2: (Limit point compact T_1 is countably compact)

A limit point compact T_1-space is countably compact.

Proof

Let $X = \bigcup U_i$ be a countable cover. If possibly, suppose there is no finite subcover. In particular, $X \setminus \bigcup_{i=1}^n U_i \neq \emptyset$ for each $n \geq 1$. Moreover, $X \setminus \bigcup_{i=1}^n U_i \neq \emptyset$ must be infinite, otherwise we can readily get a finite sub-cover. Inductively choose $x_n \notin \bigcup_{i=1}^n U_i \cup \{x_1, \ldots, x_{n-1}\}$. Thus, we have an infinite set $A = \{x_i\}$, which admits a limit point, say, x. Since X is T_1, it follows that for any open nbd $x \in U \subset X$, we must have $A \cap (U \setminus \{x\})$ is infinite (Check!). Now, we have $x \in U_{i_0}$ for some i_0. But by construction, U_{i_0} contains at most finitely many x_i, a contradiction. Hence, we must have a finite subcover. Thus, X is countably compact. □

Proposition 12.3: (Countably compact first countable is sequentially compact)

A first countable, countably compact space is sequentially compact.

Proof

Suppose, $\{x_n\}$ is a sequence. WLOG, assume element is distinct. If possible, suppose $A = \{x_n\}$ has no convergent subsequence.

If possible, $A = \{x_n\}$ has no convergent subsequence. Since X is first countable, for any $x \in X$, we must have some open set $x \in U_x \subset X$ such that $U_x \cap A$ is finite (Check!). Now, for any finite subset, $F \subset A$, consider the open set

$$\mathcal{O}_F := \bigcup \{U_x \mid U_x \cap A = F\}.$$

Since A is countable, there are countable finite subsets of F. Thus, $\mathcal{O} := \{\mathcal{O}_F \mid F \subset A \text{ is finite}\}$ is a countable collection, which is clearly an open cover. By countable compactness, we have a finite subcover $X = \bigcup_{i=1}^k \mathcal{O}_{F_i}$. Consider $F = \cup_{i=1}^k F_i$, which is again finite. Pick some $x_{i_0} \in A \setminus F$. Now, $\mathcal{O}_{F_i} \cap A = F_i \Rightarrow x_{i_0} \notin \cup_{i=1}^k F_i = \bigcup_{i=1}^k \mathcal{O}_{F_i} \cap A = X \cap A = A$, a contradiction. Hence, $\{x_n\}$

must have a convergent subsequence. Thus, X is sequentially compact. □

Proposition 12.4: (Limit point compact, T_1, first countable is sequentially compact)

Suppose X is a first countable, T_1, limit point compact space. Then X is sequentially compact.

Proof

Since X is limit point compact and T_1, we have X is countably compact. Since X is countably compact and first countable, we have X is sequentially compact. □

Example 12.5: (Necessity of T_1)

Recall the topolgoy $\mathcal{T}_{\rightarrow} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\}$ on $\mathbb{R}$. For any infinite subset $A \subset \mathbb{R}$, choose any x such that $x < a$ for some $a \in A$. Then, x is a limit point of A. Also, for any $x \in \mathbb{R}$, we have a countable neighborhood basis $\left\{U_i = (x - \frac{1}{n}, \infty) \,\middle|\, n \geq 1\right\}$. We have seen that $(\mathbb{R}, \mathcal{T}_{\rightarrow})$ is not T_1. Finally, observe that the sequence $\{x_n = -n\}$ has no convergent subsequence.

Definition 12.6: (Second countable)

A space X is called *second countable* if it admits a countable basis.

Definition 12.7: (Lindelöf)

A space X is called *Lindelöf* if every open cover admits a countable sub-cover.

Proposition 12.8: (Second countable is Lindelöf)

A second countable space is Lindelöf.

Proof

Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ is an open cover. Fix a countable base $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$. Suppose $J \subset \mathbb{N}$ is the subset of indices for which B_i is contained in some $U_\alpha \in \mathcal{U}$. For each B_j with $j \in J$, fix some $U_{\alpha_j} \in \mathcal{U}$ with $B_j \subset U_{\alpha_j}$. Clearly $\left\{U_{\alpha_j}\right\}_{j \in J}$ is a countable collection. For any $x \in X$, we have $x \in U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Now, there is some basic open set $x \in B_{i_0} \subset U_\alpha$. But then $x \in B_{i_0} \subset U_{\alpha_{i_0}}$. Thus, $\left\{U_{\alpha_j}\right\}_{j \in J}$ is a countable open cover, showing that X is Lindelöf. □

Proposition 12.9: (Limit point compact, Lindelöf, T_1 is compact)

A limit point compact, T_1, Lindelöf space is compact.

Proof

A limit point compact T_1 space is countably compact. A countably compact Lindelöf space is compact. □

Remark 12.10

We have observed the implications