Topology Course Notes (KSM1C03)

Day 11 : 16th September, 2025

`sequential compactness -- limit point compactness -- first countability`

11.1 Sequential and limit point compactness

Definition 11.1: (Sequentially compact)

A space X is called *sequentially compact* if every sequence $\{x_n\}$ has a convergent subsequence. A subset $Y \subset X$ is sequentially compact if every sequence $\{y_n\}$ in Y has a subsequence, that converges to some $y \in Y$.

Theorem 11.2: (Sequentially compact is equivalent to compact in metric space)

Suppose (X, d) is a metric space. Then, $Y \subset X$ is sequentially compact if and only if Y is compact.

Proof

Suppose Y is compact. Then, Y is closed and bounded. Consider a sequence $\{x_n\}$ in Y. If possible, suppose $\{x_n\}$ has no convergent subsequence in Y. Then, $\{x_n\}$ is an infinite sequence (i.e., there are infinitely many distinct elements). Now, for each $y \in Y$, there exists a ball $y \in B_y = B_d(y, \delta_y) \subset X$ such that B_y contains at most finitely many $\{x_n\}$ (as no subsequence of $\{x_n\}$ converge to y). We have $Y \subset \bigcup_{y \in Y} B_y$, which admits a finite subcover, say, $Y \subset \bigcup_{i=1}^n B_{y_i}$. But this implies Y contains at most finitely many $\{x_n\}$, which is a contradiction.

Conversely, suppose every sequence in Y has a subsequence converging in Y. Consider an open cover $\mathcal{U} = \{U_\alpha\}$ of Y by opens of X.

- Let us first show that for any $\delta > 0$, the collection $\{B_d(a, \delta) \mid a \in A\}$ has a finite sub-cover. Suppose not. Then, there is $x_1 \in A$ such that $A \not\subset B_d(x_1, \delta)$. Pick $x_2 \in A \setminus B_d(x_1, \delta)$. Then, $A \not\subset B_d(x_1, \delta) \cup B_d(x_2, \delta)$. Inductively, we have a sequence $\{x_n\}$ in A. Now, by construction, $d(x_i, x_j) \geq \delta$ for all $i \neq j$. Consequently, $\{x_n\}$ has no convergent subsequence, a contradiction. Indeed, if $x_{n_k} \to x \in A$, then $d(x_{n_k}, x) < \frac{\delta}{2}$ for all $k \geq N$. But then, $d(x_{n_{k_1}}, x_{n_{k_2}}) < \delta$ for any $k_1 \neq k_2 \geq N$.

- Next we claim that there exists a $\delta > 0$ such that for any $y \in Y$, we have $B_d(y, \delta) \subset U_\alpha$ for some α. Suppose not. Then, for each $n \geq 1$, there exists some $y_n \in Y$ such that $B_d(y_n, \frac{1}{n}) \not\subset U_\alpha$ for each α. Passing to a subsequence, we have $y_n \to y_0 \in A$. Now, $y_0 \in V_\alpha$ for some α, and so, $y_0 \in B_d(y_0, \epsilon) \subset V_\alpha$. There exists some $N_1 \geq 1$ such that $y_n \in B_d(y_0, \frac{\epsilon}{2})$

for all $n \geq N_1$. Also, there is $N_2 \geq 1$ such that $\frac{1}{N_2} < \frac{\epsilon}{2}$. Then, for any $n \geq \max\{N_1, N_2\}$, and for any $d(y_n, y) < \frac{1}{n}$ we have,

$$d(y_0, y) \leq d(y_0, y_n) + d(y_n, y) < \epsilon.$$

Thus, $B_d(y_n, \frac{1}{n}) \subset B_d(y_0, \epsilon) \subset V_\alpha$ for all $n \geq \max\{N_1, N_2\}$, a contradiction.

- Finally, pick the δ from the last step. Then, we have a cover $A \subset \bigcup_{i=1}^n B_d(x_i, \delta)$ with $x_i \in A$. But each of these balls are contained in some V_{α_i}. So, we have $A \subset \bigcup_{i=1}^n V_{\alpha_i}$.

□

Definition 11.3: (Limit point compactness)

A space X is called *limit point compact* (or *weakly countably compact*) if every infinite set $A \subset X$ has a limit point in X

Exercise 11.4: (Sequential compact implies limit point compact)

Show that a sequentially compact space is limit point compact.

Proposition 11.5: (Compact implies limit point compact)

A compact space is limit point compact.

Proof

Suppose X is a compact space which is not limit point compact. Then, there exists an infinite set A which has no limit point. In particular, A is closed, as it contains all of its limit points (which are none). Also, for every $x \in X$, there is an open set $x \in U_x \subset X$ such that $A \cap (U_x \setminus \{x\}) = \emptyset$. Observe that we have a covering $X = (X \setminus A) \cup \bigcup_{x \in A} U_x$, which admits a finite subcover, say, $X = (X \setminus A) \cup \bigcup_{i=1}^n U_{x_i}$. Now, $A \subset \bigcup_{i=1}^n U_{x_i}$. But this implies A is finite, as $A \cap U_{x_i} \setminus \{x_i\} = \emptyset$. This is a contradiction. □

Example 11.6: (Limit point comact but neither compact nor sequentially compact)

Consider the space $X = \mathbb{N} \times \{0, 1\}$, where give $\mathbb{N}$ the discrete topology, and $\{0, 1\}$ the indiscrete topology. Consider the sequence $x_n = (n, 0)$. Then, it does not have a convergent subsequence (otherwise, the first component projection will give convergent subsequence, as continuity implies sequential continuity). Also, X is not compact either, as the open cover $U_n = \{(n, 0), (n, 1)\}$ has no finite subcover. On the other hand, X is limit point compact. Indeed, say $A \subset X$ is infinite, and, without loss of generality, pick some $(a, 0) \in A$. Then, check that $(a, 1)$ is a limit point of A. Indeed, any open set containing $(a, 1)$ contains the open set $\{(a, 0), (a, 1)\}$, which obviously intersects A in a different point $(a, 0)$.

Definition 11.7: (First countable)

Given $x \in X$, a *neighborhood basis* is a collection $\{U_\alpha\}$ of open neighborhoods of x such that given any open neighborhood $x \in U \subset X$, there exists some U_α such that $x \in U_\alpha \subset U$. We say X is *first countable at x* if there exists a countable neighborhood basis $\{U_i\}$ of x. The space X is called *first countable* if it is first countable at every point.

Remark 11.8: (Decreasing neighborhood basis)

Suppose $\{U_i\}$ is a countable neighborhood basis of $x \in X$. Set $V_1 = U_1, V_2 = U_1 \cap U_2, \dots, V_j = V_{j-1} \cap U_j = \bigcap_{i=1}^{j} U_j$. Clearly, we have

$$V_1 \supset V_2 \supset \cdots \ni x.$$

We claim that $\{V_j\}$ is a neighborhood basis of x as well. Let $x \in U \subset X$ be an open neighborhood. Then, there is some $x \in U_j \subset U$. But then $x \in V_j \subset U_j \subset U$ as well. Thus, we can always assume that a countable neighborhood basis is decreasing. Note : in a discrete space $\{U_n = \{x\}\}$ is a non-strictly decreasing countable neighborhood basis of x.

Example 11.9: (Metric space is first countable)

Any metric space (X, d) is first countable. The converse is evidently not true, as any indiscrete space is also first countable.

Proposition 11.10: (Compact first countable is sequentially compact)

Suppose X is a first countable compact space. Then X is sequentially compact.

Proof

Let $\{x_n\}$ be a sequence in X with no convergent subsequence. Then $\{x_n\}$ must be an infinite set. Without loss of generality, assume each x_n are distinct (just extract such a subsequence). For each $x \in X$, fix some neighborhood basis $\mathcal{U}_x$. Now, since no subsequence of $\{x_n\}$ converges to x, there must be some $U_x \in \mathcal{U}_i^x$ such that only finitely many $\{x_n\}$ is contained in U_x. Otherwise, using the countability of $\mathcal{U}_x$, we can extract a subsequence converging to x. Now, we have a cover $X = \cup_{x \in X} U_x$, which admits a finite subcover, say, $X = \cup_{i=1}^{n} U_{x_i}$. But this implies the sequence $\{x_n\}$ is finite, a contradiction. □