Topology Course Notes (KSM1C03)

Day 10 : 11th September, 2025

`compactness -- finite product of compact`

10.1 Compactness (cont.)

Theorem 10.1: (Image of compact space)

$f : X \to Y$ be a continuous map. If X is compact, then $f(X)$ is compact.

Proof

Consider an open cover $\mathcal{V} = \{V_\alpha\}$ of $f(X)$ by opens of Y. Then, $\mathcal{U} = \{U_\alpha := f^{-1}(V_\alpha)\}$ is an open cover of X. Since X is compact, there is a finite subcover, say $X = \bigcup_{i=1}^{k} U_{\alpha_k} = \bigcup_{i=1}^{k} f^{-1}(V_{\alpha_i})$. But that, $f(X) \subset \bigcup_{i=1}^{k} V_{\alpha_i}$. Thus, $f(X)$ is compact. □

Theorem 10.2: (Maps from compact space to T_2)

Let $f : X \to Y$ be a surjective continuous map. Suppose X is compact, and Y is T_2. Then, f is an open map.

Proof

Let $U \subset X$ be an open set. Then, $C = X \setminus U$ is closed, and hence, compact. Since f is continuous, $f(C) \subset Y$ is compact. As Y is T_2, we have $f(C)$ is closed in Y. Finally, as f is surjective, we have $f(U) = Y \setminus f(X \setminus U) = Y \setminus f(C)$, which is then open. Thus, f is an open map. □

Remark 10.3: (Non-surjective map from compact to T_2)

Consider the inclusion map of the point $\{0\}$ in $\mathbb{R}$. Clearly, $\{0\}$ is compact, but the inclusion map is not open!

Exercise 10.4: (Compact to T_2 is closed)

Suppose X is compact, Y is T_2, and $f : X \to Y$ is a continuous map (not necessarily surjective). Then, show that f is a closed map.

Theorem 10.5: (Compactness of closed interval)

The closed interval $[a, b] \subset \mathbb{R}$ is compact (in the usual topology).

Proof

Suppose $\mathcal{A} = \{U_\alpha\}$ is a collection open sets of $\mathbb{R}$ covering $[a, b]$. Consider the set

$$C = \{c \in [a, b] \mid [a, c] \text{ is covered by a finite number of opens from } \mathcal{A}\}.$$

Note that $C \neq \emptyset$, since $[a, a] = \{a\}$ is clearly contained in some U_α. Let $L = \sum C$ be the least upper bound. Observe that $a \in U_\alpha \Rightarrow [a, a+\epsilon) \subset U_\alpha$ for some $e > 0$. Thus, $a < L \leq b$. Now, there is some U_β such that $L \in U_\beta$. Then, there is some $\epsilon > 0$ such that $a < L - \epsilon < L$ and $(L - \epsilon, L] \subset U_\beta$. Also, L being the least upper bound, there is some $c \in C$ such that $L - \epsilon < c < L$. Thus, $[a, c]$ is covered by finitely many opens, say, $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$. But then $[a, L] = [a, c] \cup [L - \epsilon, L]$ is covered by a finite collection $\{U_{\alpha_1}, \dots, U_{\alpha_k}, U_\beta\}$. Thus, $L \in C$.
Now, if $L < b$, then, there is some $\epsilon > 0$ such that $L < L + \epsilon < b$, and $[L, L+\epsilon] \subset U_\beta$. By a similar argument, it follows that $[a, L+\epsilon]$ is covered by finitely many opens of $\mathcal{A}$. This contradicts L be the least upper bound. Hence, $L = b$.
Thus, $[a, b]$ is covered by a finitely many sub-collection of $\mathcal{A}$. Since $\mathcal{A}$ is arbitrary, it follows that $[a, b]$ is compact. □

Exercise 10.6: (Real line is noncompact)

Show that $\mathbb{R}$ is not compact.

10.2 Product of compacts

Lemma 10.7: (Tube lemma)

Suppose Y is a compact space. Fix a point $x_0 \in X$, and suppose $W \subset X \times Y$ is an open set such that $\{x_0\} \times Y \subset$. Then, there exists an open set $x_0 \in U \subset X$ such that $\{x_0\} \times Y \subset U \times Y \subset W$.

Proof

For each $y \in Y$, consider a basic open set $(x_0, y) \in U_y \times V_y \subset W$. Now, $\{x_0\} \times Y \subset \bigcup_{y \in Y} U_y \times V_y$. Since Y, and hence $\{x_0\} \times Y$, is compact, we have a finite cover, say, $\{x_0\} \times Y \subset \bigcup_{i=1}^k U_{y_i} \times V_{y_i}$. Now, set $U = \bigcap_{i=1}^k U_{y_i}$, which is an open set with $x_0 \in U$. Clearly $\{x_0\} \times Y \subset U \times Y$. Now, for any $(x, y) \in U \times Y$, we have $(x_0, y) \in U_{y_{i_0}} \times V_{y_{i_0}}$ for some i_0. Then, $y \in V_{y_{i_0}}$. Also, $x \in U \subset U_{y_{i_0}}$. Thus, $(x, y) \in U_{y_{i_0}} \times V_{y_{i_0}}$. In other words, we have

$$\{x_0\} \times Y \subset U \times Y \subset \bigcup_{i=1}^k U_i \times V_i \subset W.$$

□

Theorem 10.8: (Finite product of compacts are compact)

If X, Y are compact, then so is $X \times Y$.

Proof

Suppose $\mathcal{W}$ is an open cover of $X \times Y$. For each $x \in X$, the space $\{x\} \times Y$ is compact, and hence, can be covered by a finite collection, say

$$\{x\} \times Y \subset \bigcup_{i=1}^{k_x} W_{x,i},$$

for $W_{x,i} \in \mathcal{W}$. Then, by the tube lemma, there exists some $x \in U_x \subset X$ such that

$$\{x\} \times Y \subset U_x \times Y \subset \bigcup_{i=1}^{k_x} W_{x,i}.$$

Now, $\{U_x\}$ is an open cover of X, which is also compact. Hence, we have a finite cover, say, $X = \bigcup_{i=1}^{n} U_{x_i}$. Then, clearly,

$$X \times Y = \bigcup_{i=1}^{n} U_{x_i} \times Y \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_{x_i}} W_{x_i,j}.$$

Thus, $X \times Y$ can be covered by finitely many elements of $\mathcal{W}$. Hence, $X \times Y$ is compact. $\square$