Topology Course Notes (KSM1C03)

Day 9 : 10^{th} September, 2025

`path connectedness -- path component -- locally connected -- locally path connected -- compactness`

9.1 Path connectedness (cont.)

Proposition 9.1: (Image of path connected set)

Let $f : X \to Y$ be continuous. Then, for any path connected subset $A \subset X$, we have $f(A) \subset Y$ path connected. In particular, if X is path connected, then so is $f(X)$.

Proof

Pick $x, y \in f(A)$. Then, $x = f(a)$ and $y = f(b)$ for some $a, b \in A$. Get a path $\gamma : [0,1] \to A$ joining a to b. Then, $h = f \circ \gamma : [0,1] \to f(A)$ is a path in $f(A)$ joining x to y. Thus, $f(A)$ is path connected. □

Exercise 9.2: (Product of path connected)

Let $\{X_\alpha\}$ be a family of path connected spaces. Show that the product space $X = \Pi X_\alpha$ is path connected. Give an example to show that X may not be path connected equipped with the box topology.

Definition 9.3: (Path component)

Given $x \in X$, the *path component* of X containing x is the largest possible path connected set of X containing x.

Proposition 9.4: (Existence of path component)

Given $x \in X$, the path component of X can be defined as

$$\mathcal{P}(x) := \{y \in X \mid \text{there is a path } f : [0,1] \to X \text{ with } f(0) = x \text{ and } f(1) = y\}.$$

Equivalently,

$$\mathcal{P}(x) := \bigcup \{P \subset X \mid x \in P,\ P \text{ is path connected}\}.$$

Proof

Let us check the first part. Firstly, note that $\mathcal{P}(x)$ is path connected. Indeed, given any two $y, z \in \mathcal{P}(x)$, we have two paths $f : [0,1] \to \mathcal{P}(x)$ and $g : [0,1] \to \mathcal{P}(x)$ joining, respectively, x to

y and x to z. We can construct the concatenated path h as follows

$$h(t) = \begin{cases} f(1-2t), & 0 \le t \le \frac{1}{2}, \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Check that h is continuous! Clearly, h is a then a path connecting y to z. Thus, $\mathcal{P}(x)$ is path connected.

Now, suppose A is the union of all path connected sets of X containing x. For any $y, z \in A$, we have $y \in P$ and $z \in Q$ for some path connected sets $x \in P, Q \subset X$. Then, we can get a path joining y to x and then from x to z, which is in $P \cup Q \subset A$. Thus, A is path connected, which is clearly the larges such set containing x. Hence, the second definition of $\mathcal{P}(x)$ is also true. □

Exercise 9.5: (Path component equivalence relation)

Define a relation $x \sim y$ if and only if x, y are in the same path component. Check that $\sim$ is an equivalence relation, and the equivalence classes are precisely the path components of X.

9.2 Locally connected and locally path connected spaces

Definition 9.6: (Locally connected)

A space X is called *locally connected at $x \in X$* if given any open neighborhood $x \in U$, there exists a (possibly smaller) open neighborhood $x \in V \subset U$, such that V is connected. The space is called *locally connected* if it is locally connected at every point $x \in X$.

Theorem 9.7

A space X is locally connected if and only if for all open set $U \subset X$, all the components of U are open.

Proof

Suppose X is locally connected. Pick some $U \subset X$ open, and a component $C \subset U$. Now, for any $x \in C \subset U$, by local connectedness, there is a connected open set $x \in V \subset U$. Since $x \in V \cap C$, we see that $V \cup C$ is connected. But C is the larges connected set containing x. Thus, $x \in V \subset C$, proving that $x \in \mathring{C}$. Thus, C is open.

Conversely, suppose for any open $U \subset X$, each component of U is open. Fix some x and some open neighborhood $x \in U$. Consider the component of x in U to be C. Then, C is open. Hence, X is locally connected. □

Definition 9.8: (Locally path connected)

A space X is called *locally path connected at $x \in X$* if given any open neighborhood $x \in U$, there exists a (possibly smaller) open neighborhood $x \in V \subset U$, such that V is path connected. The space is called *locally path connected* if it is locally path connected at every point $x \in X$.

Theorem 9.9

A space X is locally path connected if and only if for all open set $U \subset X$, all the path components of U are open.

Theorem 9.10

The path components of X lies in a single component. If X is locally path connected, then the path components and the components coincide.

Proof

Suppose P is a path component, which is path connected, and hence, connected. But then P can only lie in a single component.

Suppose X is locally path connected. Then, every path components of X is open. Let C be a component. For some $x \in C$, consider P to be the path component of x. Then, $x \in P \subset C$. If $P \neq C$, then consider Q to be the union of every other path components of points of $C \setminus P$. Again, we have $Q \subset C$. Now, we have a separation $C = P \sqcup Q$ by nontrivial open sets, which contradicts the fact that C is connected. Hence, $P = C$. Thus, path components of X coincide with the components. □

9.3 Compactness

Definition 9.11: (Covering)

Given a set X, a collection $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of X is called a *covering* of X if we have $X = \bigcup_{A \in \mathcal{A}} A$. Given a topological space $(X, \mathcal{T})$, we say $\mathcal{A}$ is an *open cover* (of X) $\mathcal{A}$ is a covering of X and if each $A \in \mathcal{A}$ is an open set. A *sub-cover* of $\mathcal{A}$ is a sub-collection $\mathcal{B} \subset \mathcal{A}$, which is again a covering, i.e, $X = \bigcup_{B \in \mathcal{B}} B$.

Definition 9.12: (Compact space)

A space X is called *compact* if every open cover of X has a finite sub-cover. A subset $C \subset X$ is called compact if C is compact as a subspace.

Example 9.13: (Finite space is compact)

Any finite topological space is compact, since there can be at most finitely many open sets in X. An infinite discrete space is not compact.

Proposition 9.14: (Compact subspace)

A subset $C \subset X$ is compact if and only if given any collection $\mathcal{A} = \{A_\alpha\}$ of open sets of X, with $C \subset \bigcup A_\alpha$, we have a finite sub-collection $\{A_{\alpha_1}, \dots, A_{\alpha_k}\}$ such that $C \subset \bigcup_{i=1}^k A_{\alpha_i}$.

Proof

Suppose C is compact (as a subspace). Consider a cover $\mathcal{A} = \{A_\alpha\}$ of C by opens of X. Then, $\mathcal{A}' = \{A_\alpha \cap C\}$ is an open cover of C in the subspace topology. Since C is compact, we have a finite sub-cover, say, $\{A_{\alpha_1} \cap C, \dots, A_{\alpha_k} \cap C\}$. But then $C \subset \bigcup_{i=1}^k \mathcal{A}_{\alpha_i}$.

Conversely, suppose given any cover of C by open sets of X, we have a finite sub-cover. Choose any open cover of C (in the subspace topology), say, $\mathcal{U} = \{U_\alpha \subset C\}$. Now, each $U_\alpha = C \cap V_\alpha$ for some open $V_\alpha \subset X$. Then, $C \subset \bigcup V_\alpha$ is a cover, which has finite sub-cover, $C \subset \bigcup_{i=1}^k V_{\alpha_i}$. Clearly, $C = \bigcup_{i=1}^k C \cap V_{\alpha_i} = \bigcup_{i=1}^k U_{\alpha_i}$. Thus, C is compact. □

Exercise 9.15: (Compactness is independent of subspace)

Let $Y \subset X$ be a subspace. A subset $C \subset Y$ is compact if and only if C is compact as a subspace of X.

Proposition 9.16: (Closed in compact is compact)

Suppose X is a compact space, and $C \subset X$ is closed. Then, C is compact.

Proof

Fix some cover $\{U_\alpha\}$ of C by open sets $U_\alpha \subset X$. Now, C being closed, we have $V := X \setminus C$ is open. We have, $X = V \cup \bigcup U_\alpha$. Since X is compact, there is a finite subcover. Without loss of generality, $X = V \cup \bigcup_{i=1}^k U_{\alpha_i}$. Then, $C \subset \bigcup_{i=1}^k U_{\alpha_i}$. Hence, C is compact. □

Example 9.17: (Compact need not be closed)

Let X be an indiscrete space. Then, any subset is compact, but there are non-closed subsets.

Proposition 9.18: (Compact in T_2 is closed)

Let X be a T_2 space. Then, any compact $C \subset X$ is closed.

Proof

If $C = X$, then there is nothing to show. Otherwise, we show that any $y \in X \setminus C$ is an interior point. For each $c \in C$, by T_2, there is some open neighborhoods $y \in U_c, c \in V_c$, such that $U_c \cap V_c = \emptyset$. Now, $C \subset \bigcup_{c \in C} V_c$. Since C is compact, there are finitely many points, $c_1, \ldots, c_k$, such that

$$C \subset \bigcup_{i=1}^k V_{c_i}.$$

Let us consider $U := \bigcap_{i=1}^k U_{c_i}$, which is an open neighborhood of y. Also, $U \cap \left(\bigcup_{i=1}^k V_{c_i}\right) = \emptyset \Rightarrow U \cap C = \emptyset \Rightarrow U \subset X \setminus C$. Thus, $y \in \operatorname{int}(X \setminus C)$. Since y was arbitrary, C is closed. □

Example 9.19: (Compact is not closed in T_1)

Let X be an infinite set, equipped with the cofinite topology. Then, X is T_1, but not T_2.
Let $C = X \setminus \{x_0\}$ for some $x_0 \in X$, which is clearly not closed.
Suppose $C \subset \bigcup_{\alpha \in I} U_\alpha$ is some open covering. Choose some U_{α_0}. Now, $U_{\alpha_0} = X \setminus \{x_1, \ldots, x_k\}$ (if $U_{\alpha_0} = X$, then there is nothing to show). For each $1 \le i \le k$ with $x_i \in C$, choose some U_{α_i} such that $x_i \in U_{\alpha_i}$. If $x_i \notin C$, choose U_{α_i} arbitrary. Then, $C \subset \bigcup_{i=0}^k U_{\alpha_i}$. Thus, C is compact, but not closed.