Topology Course Notes (KSM1C03)

Day 8 : 9th September, 2025

`path connectedness`

8.1 Path connectedness

Definition 8.1: (Path connected space)

A space X is called *path connected* if for any $x, y \in X$, there exists a continuous map $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. Such an f is called a *path* joining x to y. A subset $P \subset X$ is called a *path connected set* if P is path connected in the subspace topology.

Exercise 8.2: (Path connected set)

Check that $P \subset X$ is a path connected set if and only if for any $x, y \in P$, there exists a path $\gamma : [0,1] \to X$ joining $x = \gamma(0)$ to $y = \gamma(1)$, such that γ is contained in P.

Exercise 8.3: (Star connected spaces are path connected)

Given a space X and fixed point $x_0 \in X$, suppose for any $x \in X$ there exists a path in X joining x_0 to x. Show that X is path connected. What about the converse?

Proposition 8.4: (Path connected spaces are connected)

If X is a path connected space, then X is connected.

Proof

Suppose not. Then, there is a continuous surjection $g : X \to \{0,1\}$. Pick $x \in g^{-1}(0)$ and $y \in g^{-1}(1)$. Get a path $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Then, $h := g \circ f : [0,1] \to 0, 1$ is a continuous surjection, which contradicts the connectivity of $[0,1]$. Hence, X is connected. ☐

Proposition 8.5: (Connected open sets of $\mathbb{R}^n$ are path connected)

Connected open sets of $\mathbb{R}^n$ are path connected.

Proof

Let U be a connected open subset of $\mathbb{R}^n$. If $U = \emptyset$, there is nothing to show. Fix some $x \in u$. Consider the subset

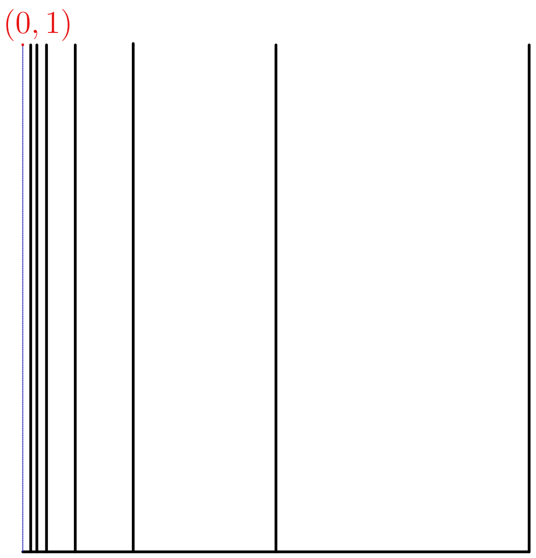
$$A = \{y \in U \mid \text{there is path in } U \text{ from } x \text{ to } y\}.$$

Clearly $A \neq \emptyset$ as $x \in A$.

Let us show A is open. Say, $y \in A$. Then, there exists a Euclidean ball $y \in B(y, \epsilon) \subset U$. Now, it is clear that for any $z \in B(y, \epsilon)$ the radial line joining y to z is a path, contained in $B(y, \epsilon)$, and hence, in U. Thus, by concatenating, we get a path from x to any $z \in B(y, \epsilon)$, showing $B(y, \epsilon) \subset A$. Thus, A is open.
Next, we show that A is closed. Let $y \in U$ be an adherent point of A. As U is open, we get some ball $y \in B(y, \epsilon) \subset U$. Now, $B(y, \epsilon) \cap A \neq \emptyset$. Say, $z \in B(y, \epsilon) \cap A$. Then, we can get a path from x to y by first getting a path to z (which exists, since $z \in A$), and then considering the radial line from z to y. Clearly, this path is contained in U. Thus, $y \in A$. Hence, A is closed.
But U is connected. Hence, the only non-empty clopen set of U is U. That is, $A = U$. But then clearly U is path connected. ☐

In general, connected spaces need not be path connected! Here is one such example. Consider $K_0 := \left\{ \frac{1}{n} \;\middle|\; n \geq 1 \right\}$, and the set

$$C := ([0,1] \times \{0\}) \cup (K_0 \times [0,1]) \subset \mathbb{R}.$$



Comb space. Removing the dotted blue line $\{0\} \times (0,1)$, we get the deleted comb space.

In the picture, this is the collection of vertical black lines, along with the 'spine' $[0,1]$ along the x-axis. It is easy to see that C is path connected, and hence, connected. Indeed, any point can be joined by a path to the origin $(0,0)$. The closure of C in $\mathbb{R}^2$ is called the *comb space*. One can easily see that

$$\bar{C} := C \cup (\{0\} \times [0,1]).$$

The *deleted comb space* D is obtained by removing the segment $\{0\} \times (0,1)$ from the comb space.

Theorem 8.6: (Deleted comb space is connected but not path connected)

The deleted comb space is connected, but not path connected.

Proof

Since C is connected, and $C \subset D \subset \bar{C}$, we have both the comb space and the deleted comb space are connected.

Intuitively, it is clear that there cannot be a path from $p = (0, 1) \in D$ to any other point of D. Let us prove this formally. If possible, suppose $f : [0, 1] \to D$ is a path from p to some point in D. Consider the set

$$A \coloneqq \{t \mid f(t) = p\} = f^{-1}(p).$$

Clearly, A is closed in $[0, 1]$, and it is non-empty as $0 \in A$.

Let us show that A is open. Let $t_0 \in A$. Since f is continuous, there exist some $\epsilon > 0$ such that for any $t \in [0, 1]$ with $|t - t_0| < \epsilon$, we have $\|f(t) - f(t_0)\| < \frac{1}{2}$. In particular, such $f(t)$ does not intersect the x-axis. Consider $B = \left\{x \in \mathbb{R}^2 \;\middle|\; \|x - p\| < \frac{1}{2}\right\} \cap \bar{C}$, and denote the interval

$$J = (t_0 - \epsilon, t_0 + \epsilon) \cap [0, 1].$$

Consider the first-component projection map $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, which is continuous. Observe that π_1 restricts to the continuous map $\pi : B \to K_0 \cup \{0\}$ (this is where we are using the fact B does not intersect the x-axis). Now, $h \coloneqq \pi \circ f|_J : J \to K_0 \cup \{0\}$ is a continuous map. We have $K_0 \cup \{0\}$ is totally disconnected, i.e, the only components are singletons. Now, $h(t_0) = \pi\left(f(t_0)\right) = \pi(p) = 0$. Hence, we must have $h(t) = 0$ for all $t \in J$, as J is connected and continuous image of a connected set is again connected. But then, $f(t) \in \pi^{-1}(0) = \{p\}$ for any $t \in J$, i.e, $f(t) = p$ for all $t \in J$. This shows that t_0 is an interior point of A. Thus, A is open.

Since $[0, 1]$ is connected, we must have $A = [0, 1]$, as it is a nonempty clopen set. But then the original path f is constant at p. Since f was an arbitrary path from p, we see that D is not path connected. □

Remark 8.7

The above argument is a very common method of proving many statements in analysis and topology. So try to understand it thoroughly!