Class Test 2 (Solution)

2[nd] September, 2025

Q1. Given any space X, consider the equivalence relation : $x \sim y$ if and only if x and y are in the same connected component.

a) $X/_\sim$ is a T_1 space.

Proof: For any $x \in X/_\sim$, we have $q^{-1}(x)$ is a connected component of X, which is closed. But the $\{x\}$ is closed in the quotient topology. So, $X/_\sim$ is T_1.

b) $X/_\sim$ is a T_2 space.

Counterexample: Consider the space

$$X = \{p = (0,0),\ q = (0,1)\} \cup \bigcup_{n\geq 1} \left\{\frac{1}{n}\right\} \times [0,1] \subset \mathbb{R}^2.$$

The connected components of X are

$$\{p\}\,,\ \{q\}\,,\ \left\{\frac{1}{n}\right\} \times [0,1], n \geq 1.$$

Now, for any *saturated* open sets $p \in U, q \in V$, it follows that infinitely many $\left\{\frac{1}{n}\right\} \times [0,1]$ are in the intersection $U \cap V$. Consequently, the equivalence classes $[\{p\}], [\{q\}] \in X/_\sim$ cannot be separated by disjoint open sets. Hence, $X/_\sim$ is not T_2.

Alternative counterexample: Consider the space

$$Y = \{a, b, x_1, x_2, \dots\}\,,$$

with the topology

$$\begin{aligned}\mathcal{T} \coloneqq \{\emptyset, Y\} &\cup \mathcal{P}(\{x_i\}_{i=1}^\infty) \\ &\{\{a\} \cup A \mid A \subset \{x_i\}_{i=1}^\infty \text{ is cofinite}\} \\ &\{\{b\} \cup B \mid B \subset \{x_i\}_{i=1}^\infty \text{ is cofinite }\} \\ &\{\{a,b\} \cup C \mid C \subset \{x_i\}_{i=1}^\infty \text{ is cofinite }\}\end{aligned}$$

Then, $(Y, \mathcal{T})$ is totally disconnected. Now, for the equivalence classes $[a]$ and $[b]$, observe that for any two open sets $[a] \in U \subset Y/_\sim, [b] \in V \subset Y/_\sim$, we have $q^{-1}(U) \cap q^{-1}(V) \neq \emptyset$ (in fact there are infinitely many elements). Hence, $U \cap V \neq \emptyset$. So $Y/\sim$ is not T_2.

c) If $X/_\sim$ is a discrete space, then X has finitely many connected components.

Counterexample: Consider X to be any infinite discrete space. Then, X is totally disconnected, and thus, have infinitely many components. Clearly, $X/_\sim \cong X$ is again a discrete space.

d) If $X/_\sim$ is an indiscrete space, then X is connected.

Proof: Since $X/_\sim$ is T_1, the only possibility is that $X/_\sim$ is a singleton. But then X is connected.

e) If $X/_\sim$ is a connected space, then X is connected.

Proof: Suppose $X/_\sim$ is connected but X is disconnected. We then have a surjective continuous map $f : X \to \{0, 1\}$. Now, for any connected component C, we must have $C \subset f^{-1}(0)$ or $C \subset f^{-1}(1)$. Then, we have a well-defined induced map $\tilde{f} : X \to \{0, 1\}$, satisfying $\tilde{f} \circ q = f$, which is continuous by the property of the quotient topology. Since f is surjective, so is $\tilde{f}$. But this contradicts that $X/_\sim$ is connected. Hence, X must be connected.

f) If X is totally disconnected, then the quotient map $q : X \to X/_\sim$ is a homeomorphism.

Proof: By the definition of totally disconnected, it follows that q is a bijection. q is a continuous map, as it is a quotient map. For any $U \subset X$ open, it follows that $U = q^{-1}(q(U))$, and hence, $q(U)$ is open in $X/_\sim$. Thus, q is a homeomorphism.